Theorem 2.122. A lattice $\mathfrak{A}$ with least element 0 is co-brouwerian with pseudodifference $\setminus^*$ iff $\setminus^*$ is a binary operation on $\mathfrak{A}$ satisfying the following identities:

1. $a \setminus^* a = 0$;
2. $a \sqcup (b \setminus^* a) = a \sqcup b$;
3. $b \sqcup (b \setminus^* a) = b$;
4. $(b \sqcup c) \setminus^* a = (b \setminus^* a) \sqcup (c \setminus^* a)$.

Proof.

⇐. We have

$$c \sqsupseteq b \setminus^* a \Rightarrow c \sqcup a \sqsupseteq a \sqcup (b \setminus^* a) = a \sqcup b \sqsupseteq b;$$

$c \sqcup a \sqsupseteq b \Rightarrow c = c \sqcup (c \setminus^* a) \sqsupseteq (a \setminus^* a) \sqcup (c \setminus^* a) = (a \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a$.

So $c \sqsupseteq b \setminus^* a \Leftrightarrow c \sqcup a \sqsupseteq b$ that is $a \sqcup -$ is an upper adjoint of $- \setminus^* a$. By a theorem above our lattice is co-brouwerian. By another theorem above $\setminus^*$ is a pseudodifference.

⇒.

1. Obvious.
2. $$\begin{aligned} a \sqcup (b \setminus^* a) &= \\ a \sqcup \bigsqcap \{ z \in \mathfrak{A} \mid b \sqsubseteq a \sqcup z \} &= \\ \bigsqcap \{ a \sqcup z \mid z \in \mathfrak{A}, b \sqsubseteq a \sqcup z \} &= \\ a \sqcup b. & \end{aligned}$$
3. $b \sqcup (b \setminus^* a) = b \sqcup \bigsqcap \{ z \in \mathfrak{A} \mid b \sqsubseteq a \sqcup z \} = \bigsqcap \{ b \sqcup z \mid z \in \mathfrak{A}, b \sqsubseteq a \sqcup z \} = b$.
4. Obviously $(b \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a$ and $(b \sqcup c) \setminus^* a \sqsupseteq c \setminus^* a$. Thus $(b \sqcup c) \setminus^* a \sqsupseteq (b \setminus^* a) \sqcup (c \setminus^* a)$. We have $$\begin{aligned} (b \setminus^* a) \sqcup (c \setminus^* a) \sqcup a &= \\ ((b \setminus^* a) \sqcup a) \sqcup ((c \setminus^* a) \sqcup a) &= \\ (b \sqcup a) \sqcup (c \sqcup a) &= \\ a \sqcup b \sqcup c &\sqsupseteq \\ b \sqcup c. & \end{aligned}$$

From this by definition of adjoints: $(b \setminus^* a) \sqcup (c \setminus^* a) \sqsupseteq (b \sqcup c) \setminus^* a$. □

Theorem 2.123. $(\bigsqcup S) \setminus^* a = \bigsqcup \{ x \setminus^* a \mid x \in S \}$ for all $a \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a co-brouwerian lattice and $\bigsqcup S$ is defined.

Proof. Because lower adjoint preserves all suprema. □

Theorem 2.124. $(a \setminus^* b) \setminus^* c = a \setminus^* (b \sqcup c)$ for elements a, b, c of a complete co-brouwerian lattice.

Proof. $a \setminus^* b = \bigsqcap \{ z \in \mathfrak{A} \mid a \sqsubseteq b \sqcup z \}$.
$(a \setminus^* b) \setminus^* c = \bigsqcap \{ z \in \mathfrak{A} \mid a \setminus^* b \sqsubseteq c \sqcup z \}$.
$a \setminus^* (b \sqcup c) = \bigsqcap \{ z \in \mathfrak{A} \mid a \sqsubseteq b \sqcup c \sqcup z \}$.
It is left to prove $a \setminus^* b \sqsubseteq c \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup c \sqcup z$.
Let $a \setminus^* b \sqsubseteq c \sqcup z$. Then $a \sqcup b \sqsubseteq b \sqcup c \sqcup z$ by the lemma and consequently $a \sqsubseteq b \sqcup c \sqcup z$.
Let $a \sqsubseteq b \sqcup c \sqcup z$. Then $a \setminus^* b \sqsubseteq (b \sqcup c \sqcup z) \setminus^* b \sqsubseteq c \sqcup z$ by a theorem above. □

2.1.15 Dual pseudocomplement on co-Heyting lattices

Proposition 2.125. For co-Heyting algebras $1 \setminus^* b = b^+$.