2. Let $\mathfrak{A}$ be a complete lattice and f preserves all infima. Let

$$g(a) = \bigsqcap \{x \in \mathfrak{A} \mid fx \sqsupseteq a\}.$$

Since f preserves infima, we have

$$f(g(a)) = \bigsqcap \{f(x) \mid x \in \mathfrak{A}, fx \sqsupseteq a\} \sqsupseteq a.$$

$g(f(b)) = \bigsqcap \{x \in \mathfrak{A} \mid fx \sqsupseteq fb\} \sqsubseteq b$.

Obviously f is monotone and thus g is also monotone.

So f is the upper adjoint of g. □

Corollary 2.105. Let f be a function from a complete lattice $\mathfrak{A}$ to a poset $\mathfrak{B}$. Then:

1. f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all infima in $\mathfrak{A}$.
2. f is an lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all suprema in $\mathfrak{A}$.

2.1.14 Co-Brouwerian lattices

Definition 2.106. Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of $a \in \mathfrak{A}$ is

$$\max \{c \in \mathfrak{A} \mid c \asymp a\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

Definition 2.107. Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of $a \in \mathfrak{A}$ is

$$\min \{c \in \mathfrak{A} \mid c \equiv a\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

Proposition 2.108. If a is a complemented element of a bounded distributive lattice, then $\bar{a}$ is both pseudocomplement and dual pseudocomplement of a.

Proof. Because of duality it is enough to prove that $\bar{a}$ is pseudocomplement of a.

We need to prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$ for every element c of our poset, and $\bar{a} \asymp a$. The second is obvious. Let's prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$.

Really, let $c \asymp a$. Then $c \sqcap a = 0$; $\bar{a} \sqcup (c \sqcap a) = \bar{a}$; $(\bar{a} \sqcup c) \sqcap (\bar{a} \sqcup a) = \bar{a}$; $\bar{a} \sqcup c = \bar{a}$; $c \sqsubseteq \bar{a}$. □

Definition 2.109. Let $\mathfrak{A}$ be a join-semilattice. Let $a, b \in \mathfrak{A}$. *Pseudodifference* of a and b is

$$\min \{z \in \mathfrak{A} \mid a \sqsubseteq b \sqcup z\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.

Remark 2.110. I do not require that a^* is undefined if there are no pseudocomplement of a and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplement, and quasidifference which generalize pseudo-* counterparts. I will denote a^* the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

Obvious 2.111. Dual pseudocomplement is the dual of pseudocomplement.

Definition 2.112. *Co-brouwerian lattice* is a lattice for which pseudodifference of any two its elements is defined.

Proposition 2.113. Every non-empty co-brouwerian lattice $\mathfrak{A}$ has least element.

Proof. Let a be an arbitrary lattice element. Then

$$a \setminus^* a = \min \{z \in \mathfrak{A} \mid a \sqsubseteq a \sqcup z\} = \min \mathfrak{A}.$$