2. Similar. □

Definition 2.100. A function f is called *idempotent* iff $f(f(X)) = f(X)$ for every argument X.

Proposition 2.101. $f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.

Proof. $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

Theorem 2.102. Each of two adjoints is uniquely determined by the other.

Proof. Let p and q be both upper adjoints of f. We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \sqsubseteq p(y) \Leftrightarrow f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \sqsubseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \sqsubseteq p(y)$. So $q(y) = p(y)$. □

Theorem 2.103. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

1. Both:
   1. If f is monotone and $g(b) = \max \{x \in \mathfrak{A} \mid fx \sqsubseteq b\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f.
   2. If $g: \mathfrak{B} \rightarrow \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max \{x \in \mathfrak{A} \mid fx \sqsubseteq b\}$ for every $b \in \mathfrak{B}$.
2. Both:
   1. If f is monotone and $g(b) = \min \{x \in \mathfrak{A} \mid fx \sqsupseteq b\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f.
   2. If $g: \mathfrak{B} \rightarrow \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min \{x \in \mathfrak{A} \mid fx \sqsupseteq b\}$ for every $b \in \mathfrak{B}$.

Proof. We will prove only the first as the second is its dual.

1. Let $g(b) = \max \{x \in \mathfrak{A} \mid fx \sqsubseteq b\}$ for every $b \in \mathfrak{B}$. Then

   $$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max \{x \in \mathfrak{A} \mid fx \sqsubseteq y\} \Rightarrow fx \sqsubseteq y$$

   (because f is monotone) and

   $$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max \{x \in \mathfrak{A} \mid fx \sqsubseteq y\} \Leftarrow fx \sqsubseteq y.$$

   So $fx \sqsubseteq y \Leftrightarrow x \sqsubseteq gy$ that is f is the lower adjoint of g.
2. We have

   $$g(b) = \max \{x \in \mathfrak{A} \mid fx \sqsubseteq b\} \quad \Leftrightarrow$$
   $$fgb \sqsubseteq b \wedge \forall x \in \mathfrak{A}: (fx \sqsubseteq b \Rightarrow x \sqsubseteq gb)$$

   what is true by properties of adjoints. □

Theorem 2.104. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

1. If f is an upper adjoint, f preserves all existing infima in $\mathfrak{A}$.
2. If $\mathfrak{A}$ is a complete lattice and f preserves all infima, then f is an upper adjoint of a function $\mathfrak{B} \rightarrow \mathfrak{A}$.
3. If f is a lower adjoint, f preserves all existing suprema in $\mathfrak{A}$.
4. If $\mathfrak{A}$ is a complete lattice and f preserves all suprema, then f is a lower adjoint of a function $\mathfrak{B} \rightarrow \mathfrak{A}$.

Proof. We will prove only first two items because the rest items are similar.

1. Let $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcap S$ exists. $f \bigsqcap S$ is a lower bound for $\langle f \rangle S$ because f is order-preserving. If a is a lower bound for $\langle f \rangle S$ then $\forall x \in S: a \sqsubseteq fx$ that is $\forall x \in S: ga \sqsubseteq x$ where g is the lower adjoint of f. Thus $ga \sqsubseteq \bigsqcap S$ and hence $f \bigsqcap S \sqsupseteq a$. So $f \bigsqcap S$ is the greatest lower bound for $\langle f \rangle S$.