Remark 2.81. This definition is valid even for posets without least element.

I will denote $(\operatorname{atoms}^{\mathfrak{A}} a)$ or just $(\operatorname{atoms} a)$ the set of atoms contained in an element a of a poset $\mathfrak{A}$. I will denote $\operatorname{atoms}^{\mathfrak{A}}$ the set of all atoms of a poset $\mathfrak{A}$.

Definition 2.82. A poset $\mathfrak{A}$ is called *atomic* iff $\operatorname{atoms} a \neq \emptyset$ for every non-least element a of the poset $\mathfrak{A}$.

Definition 2.83. *Atomistic poset* is such a poset that $a = \bigsqcup \operatorname{atoms} a$ for every non-least element a of this poset.

Obvious 2.84. Every atomistic poset is atomic.

Proposition 2.85. Let $\mathfrak{A}$ be a poset. If a is an atom of $\mathfrak{A}$ and $B \in \mathfrak{A}$ then $a \sqsubseteq B \Leftrightarrow a \not\asymp B$.

Proof.

- $\Rightarrow$. $a \sqsubseteq B \Rightarrow a \sqsubseteq a \wedge a \sqsubseteq B$, thus $a \not\asymp B$ because a is not least.
- $\Leftarrow$. $a \not\asymp B$ implies existence of non-least element x such that $x \sqsubseteq B$ and $x \sqsubseteq a$. Because a is an atom, we have $x = a$. So $a \sqsubseteq B$. □

Theorem 2.86. $\operatorname{atoms} \bigsqcap S = \bigcap \langle \operatorname{atoms} \rangle S$ whenever $\bigsqcap S$ is defined for every $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset.

Proof. For any atom c

$$\begin{aligned} c \in \operatorname{atoms} \bigsqcap S &\Leftrightarrow \\ c \sqsubseteq \bigsqcap S &\Leftrightarrow \\ \forall a \in S: c \sqsubseteq a &\Leftrightarrow \\ \forall a \in S: c \in \operatorname{atoms} a &\Leftrightarrow \\ c \in \bigcap \langle \operatorname{atoms} \rangle S. & \end{aligned}$$

□

Corollary 2.87. $\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ for an arbitrary meet-semilattice.

Theorem 2.88. A complete boolean lattice is atomic iff it is atomistic.

Proof.

- $\Leftarrow$. Obvious.
- $\Rightarrow$. Let $\mathfrak{A}$ be an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigsqcup \operatorname{atoms} a \sqsubset a$. If $x \in \operatorname{atoms}(a \setminus b)$ then $x \sqsubseteq a \setminus b$ and so $x \sqsubseteq a$ and hence $x \sqsubseteq b$. But we have $x = x \sqcap b \sqsubseteq (a \setminus b) \sqcap b = 0$ what contradicts to our supposition. □

2.1.11 Kuratowski's lemma

Theorem 2.89. (Kuratowski lemma) Any chain in a poset is contained in a maximal chain (if we order chains by inclusion).

I will skip the proof of Kuratowski lemma as this proof can be found in any set theory or order theory reference.

2.1.12 Homomorphisms of posets and lattices

Definition 2.90. A *monotone* function (also called *order homomorphism*) from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$.