$x \sqcap \bigsqcup S \sqsupseteq \bigsqcup \langle x \sqcap \rangle S$ is obvious. Now let u be any upper bound of $\langle x \sqcap \rangle S$, that is $x \sqcap y \sqsubseteq u$ for all $y \in S$. Then

$$y = y \sqcap (x \sqcup \bar{x}) = (y \sqcap x) \sqcup (y \sqcap \bar{x}) \sqsubseteq u \sqcup \bar{x},$$

and so $\bigsqcup S \sqsubseteq u \sqcup \bar{x}$. Thus

$$x \sqcap \bigsqcup S \sqsubseteq x \sqcap (u \sqcup \bar{x}) = (x \sqcap u) \sqcup (x \sqcap \bar{x}) = (x \sqcap u) \sqcup 0 = x \sqcap u \sqsubseteq u,$$

that is $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle S$. □

Theorem 2.71. (infinite De Morgan's laws) For every subset S of a complete boolean lattice

1. $\overline{\bigsqcup S} = \bigsqcap_{x \in S} \bar{x}$;
2. $\overline{\bigsqcap S} = \bigsqcup_{x \in S} \bar{x}$.

Proof. It's enough to prove that $\bigsqcup S$ is a complement of $\bigsqcap_{x \in S} \bar{x}$ (the second follows from duality). Really, using the previous theorem:

$$\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \bigsqcap_{x \in S} \left\langle \bigsqcup S \sqcup \right\rangle \bar{x} = \bigsqcap_{x \in S} \left\{ \bigsqcup S \sqcup \bar{x} \mid x \in S \right\} \sqsupseteq \bigsqcap_{x \in S} \{ x \sqcup \bar{x} \mid x \in S \} = 1;$$

$$\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bigsqcup_{y \in S} \left\langle \bigsqcap_{x \in S} \bar{x} \sqcap \right\rangle y = \bigsqcup_{y \in S} \left\{ \bigsqcap_{x \in S} \bar{x} \sqcap y \mid y \in S \right\} \sqsubseteq \bigsqcup_{y \in S} \{ \bar{y} \sqcap y \mid y \in S \} = 0.$$

So $\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = 1$ and $\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = 0$. □

2.1.9 Center of a lattice

Definition 2.72. The *center* $Z(\mathfrak{A})$ of a bounded distributive lattice $\mathfrak{A}$ is the set of its complemented elements.

Remark 2.73. For a definition of center of non-distributive lattices see [5].

Remark 2.74. In [23] the word center and the notation $Z(\mathfrak{A})$ are used in a different sense.

Definition 2.75. A sublattice K of a complete lattice L is a *closed sublattice* of L if K contains the meet and the join of any its nonempty subset.

Theorem 2.76. Center of an infinitely distributive lattice is its closed sublattice.

Proof. See [16]. □

Remark 2.77. See [17] for a more strong result.

Theorem 2.78. The center of a bounded distributive lattice constitutes its sublattice.

Proof. Let $\mathfrak{A}$ be a bounded distributive lattice and $Z(\mathfrak{A})$ be its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \sqcup \bar{b}$ is the complement of $a \sqcap b$ because

$$(a \sqcap b) \sqcap (\bar{a} \sqcup \bar{b}) = (a \sqcap b \sqcap \bar{a}) \sqcup (a \sqcap b \sqcap \bar{b}) = 0 \sqcup 0 = 0 \quad \text{and}$$
$$(a \sqcap b) \sqcup (\bar{a} \sqcup \bar{b}) = (a \sqcup \bar{a} \sqcup \bar{b}) \sqcap (b \sqcup \bar{a} \sqcup \bar{b}) = 1 \sqcap 1 = 1.$$

So $a \sqcap b$ is complemented. Similarly $a \sqcup b$ is complemented. □

Theorem 2.79. The center of a bounded distributive lattice constitutes a boolean lattice.

Proof. Because it is a distributive complemented lattice. □

2.1.10 Atoms of posets

Definition 2.80. An *atom* of a poset is an element which has no non-least subelements.