Definition 2.63. An element of bounded distributive lattice is called *complemented* when its complement exists.

Definition 2.64. A distributive lattice is a *complemented lattice* iff every its element is complemented.

Proposition 2.65. For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \sqcup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.

Proof. $((a \setminus b) \setminus c) \sqcap c = 0$; $((a \setminus b) \setminus c) \sqcup c = (a \setminus b) \sqcup c$; $(a \setminus b) \sqcap b = 0$; $(a \setminus b) \sqcup b = a \sqcup b$.

We need to prove $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = 0$ and $((a \setminus b) \setminus c) \sqcup (b \sqcup c) = a \sqcup (b \sqcup c)$.

In fact,

$$\begin{aligned} ((a \setminus b) \setminus c) \sqcap (b \sqcup c) &= \\ (((a \setminus b) \setminus c) \sqcap b) \sqcup (((a \setminus b) \setminus c) \sqcap c) &= \\ (((a \setminus b) \setminus c) \sqcap b) \sqcup 0 &= \\ ((a \setminus b) \setminus c) \sqcap b &\sqsubseteq \\ (a \setminus b) \sqcap b &= 0, \end{aligned}$$

so $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = 0$;

$$\begin{aligned} ((a \setminus b) \setminus c) \sqcup (b \sqcup c) &= \\ (((a \setminus b) \setminus c) \sqcup c) \sqcup b &= \\ (a \setminus b) \sqcup c \sqcup b &= \\ ((a \setminus b) \sqcup b) \sqcup c &= \\ a \sqcup b \sqcup c. \end{aligned}$$

□

2.1.8 Boolean lattices

Definition 2.66. A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is $\mathscr{P}A$ where A is a set, ordered by set inclusion.

Theorem 2.67. (De Morgan's laws) For every elements a, b of a boolean lattice

1. $\overline{a \sqcup b} = \bar{a} \sqcap \bar{b}$;
2. $\overline{a \sqcap b} = \bar{a} \sqcup \bar{b}$.

Proof. We will prove only the first as the second is dual.

It is enough to prove that $a \sqcup b$ is a complement of $\bar{a} \sqcap \bar{b}$. Really:

$$(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) \sqsubseteq a \sqcap (\bar{a} \sqcap \bar{b}) = (a \sqcap \bar{a}) \sqcap \bar{b} = 0 \sqcap \bar{b} = 0;$$
$$(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = ((a \sqcup b) \sqcup \bar{a}) \sqcap ((a \sqcup b) \sqcup \bar{b}) \sqsupseteq (a \sqcup \bar{a}) \sqcap (b \sqcup \bar{b}) = 1 \sqcap 1 = 1.$$

Thus $(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) = 0$ and $(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = 1$. □

Definition 2.68. A complete lattice $\mathfrak{A}$ is *join infinite distributive* when $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle S$; complete lattice is *meet infinite distributive* when $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle S$ for all $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$.

Definition 2.69. *Infinite distributive complete lattice* is a complete lattice which is both join infinite distributive and meet infinite distributive.

Theorem 2.70. Every complete boolean lattice is both join infinite distributive and meet infinite distributive.

Proof. We will prove only join infinitely distributivity, as the other is dual.

Let S be a subset of a complete boolean lattice.