Proof. We will prove only the first as the second is dual.

By definition of joins, it is enough to prove $y \sqsupseteq \bigsqcup \bigcup S \Leftrightarrow y \sqsupseteq \bigsqcup \{\bigsqcup X \mid X \in S\}$.

Really, $y \sqsupseteq \bigsqcup \bigcup S \Leftrightarrow \forall x \in \bigcup S\colon y \sqsupseteq x \Leftrightarrow \forall X \in S \forall x \in X\colon y \sqsupseteq x \Leftrightarrow \forall X \in S\colon y \sqsupseteq \bigsqcup X \Leftrightarrow y \sqsupseteq \bigsqcup \{\bigsqcup X \mid X \in S\}$. □

2.1.6 Distributivity of lattices

Definition 2.53. A *distributive* lattice is such lattice $\mathfrak{A}$ that for every $x, y, z \in \mathfrak{A}$

1. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Theorem 2.54. For a lattice to be distributive it is enough just one of the conditions:

1. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Proof. $(x \sqcup y) \sqcap (x \sqcup z) = ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z) = x \sqcup ((x \sqcap z) \sqcup (y \sqcap z)) = (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) = x \sqcup (y \sqcap z)$ (applied $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ twice). □

2.1.7 Difference and complement

Definition 2.55. Let $\mathfrak{A}$ be a distributive lattice with least element 0. The *difference* (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \sqcap c = 0$ and $a \sqcup b = b \sqcup c$. I will call b *substractive* from a when $a \setminus b$ exists.

Theorem 2.56. If $\mathfrak{A}$ is a distributive lattice with least element 0, there exists no more than one difference of elements a, b.

Proof. Let c and d be both differences $a \setminus b$. Then $b \sqcap c = b \sqcap d = 0$ and $a \sqcup b = b \sqcup c = b \sqcup d$. So

$$c = c \sqcap (b \sqcup c) = c \sqcap (b \sqcup d) = (c \sqcap b) \sqcup (c \sqcap d) = 0 \sqcup (c \sqcap d) = c \sqcap d.$$

Similarly $d = d \sqcap c$. Consequently $c = c \sqcap d = d \sqcap c = d$. □

Definition 2.57. I will call b *complementive* to a iff there exists $c \in \mathfrak{A}$ such that $b \sqcap c = 0$ and $b \sqcup c = a$.

Proposition 2.58. b is complementive to a iff b is substractive from a and $b \sqsubseteq a$.

Proof.

$\Leftarrow$. Obvious.

$\Rightarrow$. We deduce $b \sqsubseteq a$ from $b \sqcup c = a$. Thus $a \sqcup b = a = b \sqcup c$. □

Proposition 2.59. If b is complementive to a then $(a \setminus b) \sqcup b = a$.

Proof. Because $b \sqsubseteq a$ by the previous proposition. □

Definition 2.60. Let $\mathfrak{A}$ be a bounded distributive lattice. The *complement* (denoted $\bar{a}$) of an element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \sqcap b = 0$ and $a \sqcup b = 1$.

Proposition 2.61. If $\mathfrak{A}$ is a bounded distributive lattice then $\bar{a} = 1 \setminus a$.

Proof. $b = \bar{a} \Leftrightarrow b \sqcap a = 0 \wedge b \sqcup a = 1 \Leftrightarrow b \sqcap a = 0 \wedge 1 \sqcup a = a \sqcup b \Leftrightarrow b = 1 \setminus a$. □

Corollary 2.62. If $\mathfrak{A}$ is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.