2. If $a \sqcap b$ exists then $y \sqsubseteq a \sqcap b \Leftrightarrow y \sqsubseteq a \wedge y \sqsubseteq b$.

2.1.4 Semilattices

Definition 2.41.

1. A *join-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcup b$ is defined for every $a, b \in \mathfrak{A}$.
2. A *meet-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcap b$ is defined for every $a, b \in \mathfrak{A}$.

Theorem 2.42.

1. The operation $\sqcup$ is associative for any join-semilattice.
2. The operation $\sqcap$ is associative for any meet-semilattice.

Proof. I will prove only the first as the second follows by duality.

We need to prove $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$ for every $a, b, c \in \mathfrak{A}$.

Taking into account the definition of join, it is enough to prove that

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c)$$

for every $x \in \mathfrak{A}$. Really, this follows from the chain of equivalences:

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup b \wedge x \sqsupseteq c \Leftrightarrow x \sqsupseteq a \wedge x \sqsupseteq b \wedge x \sqsupseteq c \Leftrightarrow x \sqsupseteq a \wedge x \sqsupseteq b \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c).$$ □

Obvious 2.43. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

Obvious 2.44. $a \equiv b$ if $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

2.1.5 Lattices and complete lattices

Definition 2.45. A *bounded* poset is a poset having both least and greatest elements.

Definition 2.46. *Lattice* is a poset which is both join-semilattice and meet-semilattice.

Definition 2.47. A *complete lattice* is a poset $\mathfrak{A}$ such that for every $X \in \mathscr{P}\mathfrak{A}$ both $\bigsqcup X$ and $\bigsqcap X$ exist.

Obvious 2.48. Every complete lattice is a lattice.

Proposition 2.49. Every complete lattice is a bounded poset.

Proof. $\bigsqcup \emptyset$ is the least and $\bigsqcap \emptyset$ is the greatest element. □

Theorem 2.50. Let $\mathfrak{A}$ be a poset.

1. If $\bigsqcup X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.
2. If $\bigsqcap X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.

Proof. See [26] or any lattice theory reference. □

Obvious 2.51. If $X \subseteq Y$ for some $X, Y \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice, then

1. $\bigsqcup X \sqsubseteq \bigsqcup Y$;
2. $\bigsqcap X \sqsupseteq \bigsqcap Y$.

Proposition 2.52. If $S \in \mathscr{P}\mathscr{P}\mathfrak{A}$ then for every complete lattice $\mathfrak{A}$

1. $\bigsqcup \bigcup S = \bigsqcup \{\bigsqcup X \mid X \in S\}$;
2. $\bigsqcap \bigcup S = \bigsqcap \{\bigsqcap X \mid X \in S\}$.