Definition 2.29.

- A *minimal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X: (a \sqsupseteq x \wedge x \neq a)$.
- A *maximal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X: (a \sqsubseteq x \wedge x \neq a)$.

Remark 2.30. Minimal element is not the same as minimum, and maximal element is not the same as maximum.

Obvious 2.31.

1. The least element (if it exists) is a minimal element.
2. The greatest element (if it exists) is a maximal element.

Exercise 2.1. Show that there may be more than one minimal and more than one maximal element for some poset.

Definition 2.32. *Upper bounds* of a set X is the set $\{y \in \mathfrak{A} \mid \forall x \in X: y \sqsupseteq x\}$.

The dual notion:

Definition 2.33. *Lower bounds* of a set X is the set $\{y \in \mathfrak{A} \mid \forall x \in X: y \sqsubseteq x\}$.

Definition 2.34. *Join* $\bigsqcup X$ (also called *supremum* and denoted "$\sup X$") of a set X is the least element of its upper bounds (if it exists).

Definition 2.35. *Meet* $\bigsqcap X$ (also called *infimum* and denoted "$\inf X$") of a set X is the greatest element of its lower bounds (if it exists).

We will write $b = \bigsqcup X$ when $b \in \mathfrak{A}$ is the join of X or say that $\bigsqcup X$ does not exist if there are no such $b \in \mathfrak{A}$. (And dually for meets.)

Exercise 2.2. Provide an example of $\bigsqcup X \notin X$ for some set X on some poset.

I will denote meets and joins for a specific poset $\mathfrak{A}$ as $\bigsqcap^{\mathfrak{A}}$ and $\bigsqcup^{\mathfrak{A}}$.

Proposition 2.36.

1. If b is the greatest element of X then $\bigsqcup X = b$.
2. If b is the least element of X then $\bigsqcap X = b$.

Proof. We will prove only the first as the second is dual.

Let b be the greatest element of X. Then upper bounds of X are $\{y \in \mathfrak{A} \mid y \sqsupseteq b\}$. Obviously b is the least element of this set, that is the join. □

Definition 2.37. *Binary joins and meets* are defined by the formulas

$$x \sqcup y = \bigsqcup \{x, y\} \quad \text{and} \quad x \sqcap y = \bigsqcap \{x, y\}.$$

Obvious 2.38. $\sqcup$ and $\sqcap$ are symmetric operations (whenever these are defined for given x and y).

Theorem 2.39.

1. If $\bigsqcup X$ exists then $y \sqsupseteq \bigsqcup X \Leftrightarrow \forall x \in X: y \sqsupseteq x$.
2. If $\bigsqcap X$ exists then $y \sqsubseteq \bigsqcap X \Leftrightarrow \forall x \in X: y \sqsubseteq x$.

Proof. I will prove only the first as the second follows by duality.

$y \sqsupseteq \bigsqcup X \Leftrightarrow y$ is an upper bound for $X \Leftrightarrow \forall x \in X: y \sqsupseteq x$. □

Corollary 2.40.

1. If $a \sqcup b$ exists then $y \sqsupseteq a \sqcup b \Leftrightarrow y \sqsupseteq a \wedge y \sqsupseteq b$.