From the above it follows that staroids on filters do not correspond (by restriction) to staroids on principal filters (or staroids on sets).

18.3. Complete staroids and multifuncoids

18.3.1. Complete free stars.

FiXme: This section should be integrated into the chapter about filters and free stars.

DEFINITION 1461. Let $\mathfrak{A}$ be a poset. *Complete free stars* on $\mathfrak{A}$ are such $S \in \mathscr{P}\mathfrak{A}$ that the least element (if it exists) is not in S and for every $T \in \mathscr{P}\mathfrak{A}$

$$\forall Z \in \mathfrak{A} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset.$$

OBVIOUS 1462. Every complete free star is a free star.

PROPOSITION 1463. $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is a complete free star iff all the following:

1°. The least element (if it exists) is not in S.
2°. $\forall Z \in \mathfrak{A} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.
3°. S is an upper set.

PROOF.

$\Rightarrow$. 1 and 2 are obvious. S is an upper set because S is a free star.

$\Leftarrow$. We need to prove that

$$\forall Z \in \mathfrak{A} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftarrow T \cap S \neq \emptyset.$$

Let $X' \in T \cap S$. Then $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq X' \Rightarrow Z \in S$ because S is an upper set.

□

PROPOSITION 1464. Let S be a complete lattice. $S \in \mathscr{P}\mathfrak{A}$ is a complete free star iff all the following:

1°. The least element (if it exists) is not in S.
2°. $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P}S$.
3°. S is an upper set.

PROOF.

$\Rightarrow$. We need to prove only $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. Let $\bigsqcup T \in S$. Because S is an upper set, we have $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq \bigsqcup T \Rightarrow Z \in S$ from which we conclude $T \cap S \neq \emptyset$.

$\Leftarrow$. We need to prove only $\forall Z \in \mathfrak{A} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.

Really, if $\forall Z \in \mathfrak{A} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S)$ then $\bigsqcup T \in S$ and thus $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$.

□

PROPOSITION 1465. Let $\mathfrak{A}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{A}$ is a complete free star iff the least element (if it exists) is not in S and for every $T \in \mathscr{P}\mathfrak{A}$

$$\bigsqcup T \in S \Leftrightarrow T \cap S \neq \emptyset.$$

PROOF.

$\Rightarrow$. We need to prove only $\bigsqcup T \in S \Leftarrow T \cap S \neq \emptyset$ what follows from that S is an upper set.

$\Leftarrow$. We need to prove only that S is an upper set. To prove this we can use the fact that S is a free star.

□