$$\forall i \in n : f_i \in \mathrm{C}''(\mu_i; \nu_i) \Leftrightarrow \forall i \in n : f_i \circ \mu_i \circ f_i^\dagger \sqsubseteq \nu_i \Rightarrow$$
$$\prod_{i\in n}^{(C)} (f_i \circ \mu_i \circ f_i^\dagger) \sqsubseteq \prod_{i\in n}^{(C)} \nu_i \Leftrightarrow \prod_{i\in n}^{(C)} f_i \circ \prod_{i\in n}^{(C)} \mu_i \circ \prod_{i\in n}^{(C)} f_i^\dagger \sqsubseteq \prod_{i\in n}^{(C)} \nu_i \Leftrightarrow$$
$$\prod_{i\in n}^{(C)} f_i \circ \prod_{i\in n}^{(C)} \mu_i \circ \left(\prod_{i\in n}^{(C)} f_i \right)^\dagger \sqsubseteq \prod_{i\in n}^{(C)} \nu_i \Leftrightarrow \prod_{i\in n}^{(C)} f_i \in \mathrm{C}''\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right).$$

□

THEOREM 1422. Let μ and ν be indexed (by some index set n) families of endofuncoids, and $f_i \in \mathsf{FCD}(\operatorname{Ob} \mu_i; \operatorname{Ob} \nu_i)$ for every $i \in n$. Then:

1°. $\forall i \in n : f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}\left(\prod^{(A)} \mu; \prod^{(A)} \nu \right)$;

2°. $\forall i \in n : f_i \in \mathrm{C}'(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}'\left(\prod^{(A)} \mu; \prod^{(A)} \nu \right)$;

3°. $\forall i \in n : f_i \in \mathrm{C}''(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}''\left(\prod^{(A)} \mu; \prod^{(A)} \nu \right)$.

PROOF. Similar to the previous theorem. □

THEOREM 1423. Let μ and ν be indexed (by some index set n) families of point-free endofuncoids between posets with least elements, and $f_i \in \mathsf{FCD}(\operatorname{Ob} \mu_i; \operatorname{Ob} \nu_i)$ for every $i \in n$. Then:

1°. $\forall i \in n : f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}\left(\prod^{(S)} \mu; \prod^{(S)} \nu \right)$;

2°. $\forall i \in n : f_i \in \mathrm{C}'(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}'\left(\prod^{(S)} \mu; \prod^{(S)} \nu \right)$;

3°. $\forall i \in n : f_i \in \mathrm{C}''(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}''\left(\prod^{(S)} \mu; \prod^{(S)} \nu \right)$.

PROOF. Similar to the previous theorem. □

17.17. Counter-examples

EXAMPLE 1424. $\upuparrows\downdownarrows f \neq f$ for some staroid f whose form is an indexed family of filters on a set.

PROOF. Let $f = \left\{ \frac{\mathcal{A} \in \mathfrak{F}(\mho)}{\uparrow \operatorname{Cor} \mathcal{A} \not\asymp \Delta} \right\}$ for some infinite set $\mho$ where Δ is some non-principal filter on $\mho$.

$$A \sqcup B \in f \Leftrightarrow \uparrow^\mho \operatorname{Cor}(A \sqcup B) \not\asymp \Delta \Leftrightarrow \uparrow^\mho \operatorname{Cor} A \sqcup \uparrow^\mho \operatorname{Cor} B \not\asymp \Delta \Leftrightarrow$$
$$\uparrow^\mho \operatorname{Cor} A \sqcap \Delta \neq \bot^{\mathfrak{F}(\mho)} \vee \uparrow^\mho \operatorname{Cor} B \sqcap \Delta \neq \bot^{\mathfrak{F}(\mho)} \Leftrightarrow A \in f \vee B \in f.$$

Obviously $\bot^{\mathfrak{F}(\mho)} \notin f$. So f is a free star. But free stars are essentially the same as 1-staroids.

$\downdownarrows f = \partial\Delta$. $\upuparrows\downdownarrows f = \star\Delta \neq f$. □

For the below counter-examples we will define a staroid ϑ with $\operatorname{arity} \vartheta = \mathbb{N}$ and $\operatorname{GR} \vartheta \in \mathscr{P}(\mathbb{N}^\mathbb{N})$ (based on a suggestion by Andreas Blass):

$$A \in \operatorname{GR} \vartheta \Leftrightarrow \sup_{i\in\mathbb{N}} \operatorname{card}(A_i \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_i \neq \emptyset.$$

PROPOSITION 1425. ϑ is a staroid.

PROOF. $(\operatorname{val} \vartheta)_i L = \mathscr{P}\mathbb{N} \setminus \{\emptyset\}$ for every $L \in (\mathscr{P}\mathbb{N})^{\mathbb{N}\setminus\{i\}}$ if

$$\sup_{i\in\mathbb{N}\setminus\{i\}} \operatorname{card}(A_j \cap j) = \mathbb{N} \wedge \forall j \in \mathbb{N} \setminus \{i\} : L_j \neq \emptyset.$$