THEOREM 1345. Cross-composition product of a family of **Rel**-morphisms is a principal funcoid.

PROOF. By the proposition and symmetry $\prod^{(C)} f$ is a pointfree funcoid. Obviously it is a funcoid $\prod_{i\in n} \operatorname{Src} f_i \to \prod_{i\in n} \operatorname{Dst} f_i$. Its completeness (and dually co-completeness) is obvious. □

17.11.5. Cross-composition product of funcoids. Let a be a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom}\mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\operatorname{Src} f_i = \mathfrak{A}_i$.

The star-composition of a with f is an anchored relation of the form $\lambda i \in \operatorname{dom}\mathfrak{A} : \operatorname{Dst} f_i$ defined by the formula

$$L \in \operatorname{GR}\operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{\mathfrak{A}_i} \forall i \in n : y_i \; [f_i] \; L_i.$$

THEOREM 1346. Let $\operatorname{Dst} f_i$ be a starrish join-semilattice for every $i \in n$.

1°. If a is a prestaroid then $\operatorname{StarComp}(a; f)$ is a staroid.

2°. If a is a completary staroid and then $\operatorname{StarComp}(a; f)$ is a completary staroid.

PROOF.

1°. First prove that $\operatorname{StarComp}(a; f)$ is a prestaroid. We need to prove that $(\operatorname{val}\operatorname{StarComp}(a; f))_j L$ (for every $j \in n$) is a free star, that is

$$\left\{ \frac{X \in (\operatorname{form} f)_j}{L \cup \{(j; X)\} \in \operatorname{GR}\operatorname{StarComp}(a; f)} \right\}$$

is a free star, that is the following is a free star

$$\left\{ \frac{X \in (\operatorname{form} f)_j}{R(X)} \right\}$$

where $R(X) \Leftrightarrow \exists y \in \prod_{i\in n} \operatorname{atoms}\mathfrak{A}_i : (\forall i \in n \setminus \{j\} : y_i \; [f_i] \; L_i \wedge y_j \; [f_j] \; X \wedge y \in \operatorname{GR} a)$.

$$R(X) \Leftrightarrow$$

$$\exists y \in \prod_{i\in n} \operatorname{atoms}^{\mathfrak{A}_i} : (\forall i \in n \setminus \{j\} : y_i \; [f_i] \; L_i \wedge y_j \; [f_j] \; X \wedge y_j \in (\operatorname{val} a)_j (y|_{n\setminus\{j\}})) \Leftrightarrow$$

$$\exists y \in \prod_{i\in n\setminus\{j\}} \operatorname{atoms}^{\mathfrak{A}_i}, y' \in \operatorname{atoms}^{\mathfrak{A}_j} : \begin{pmatrix} \forall i \in n \setminus \{j\} : y_i \; [f_i] \; L_i \wedge \\ y' \; [f_j] \; X \wedge y' \in (\operatorname{val} a)_j (y|_{n\setminus\{j\}}) \end{pmatrix} \Leftrightarrow$$

$$\exists y \in \prod_{i\in n\setminus\{j\}} \operatorname{atoms}^{\mathfrak{A}_i} \forall i \in n \setminus \{j\} : y_i \; [f_i] \; L_i \wedge$$
$$\exists y' \in \operatorname{atoms}^{\mathfrak{A}_j} : (y' \; [f_j] \; X \wedge y' \in (\operatorname{val} a)_j (y|_{n\setminus\{j\}})).$$

If $\exists y \in \prod_{i\in n\setminus\{j\}} \operatorname{atoms}^{\mathfrak{A}_i} \forall i \in n \setminus \{j\} : y_i \; [f_i] \; L_i$ is false our statement is obvious. We can assume it is true.

So it is enough to prove that

$$\left\{ \frac{X \in (\operatorname{form} f)_j}{\exists y \in \prod_{i\in n\setminus\{j\}} \operatorname{atoms}^{\mathfrak{A}_i}, y' \in \operatorname{atoms}^{\mathfrak{A}_j} : y' \; [f_j] \; X \wedge y' \in (\operatorname{val} a)_j (y|_{n\setminus\{j\}}))} \right\}$$

is a free star. That is

$$Q = \left\{ \frac{X \in (\operatorname{form} f)_j}{\exists y \in \prod_{i\in n\setminus\{j\}} \operatorname{atoms}^{\mathfrak{A}_i}, y' \in (\operatorname{atoms}^{\mathfrak{A}_j}) \cap (\operatorname{val} a)_j (y|_{n\setminus\{j\}}) : y' \; [f_j] \; X} \right\}$$

is a free star. $\perp^{(\operatorname{form} f)_j} \notin Q$ is obvious. That Q is an upper set is obvious. It remains to prove that $X_0 \sqcup X_1 \in Q \Rightarrow X_0 \in Q \vee X_1 \in Q$ for every $X_0, X_1 \in (\operatorname{form} f)_j$.