REMARK 1330. We can carry definitions (such as below defined cross-composition product) from categories with star-morphisms into plain dagger categories. This allows us to research properties of cross-composition product of indexed families of morphisms for categories with star-morphisms without separately considering the special case of dagger categories and just binary star-composition product.

17.10.1. Abrupt of quasi-invertible categories with star-morphisms.

DEFINITION 1331. The abrupt partially ordered precategory of a partially ordered precategory with star-morphisms is the abrupt precategory with the following order of morphisms:

- Indexed (by arity m for some $m \in M$) families of morphisms of C are ordered as function spaces of posets.
- Star-morphisms (which are morphisms $\mathrm{None} \to \mathrm{Obj}_m$ for some $m \in M$) are ordered in the same order as in the precategory with star-morphisms.
- Morphisms $\mathrm{None} \to \mathrm{None}$ which are only the identity morphism ordered by the unique order on this one-element set.

We need to prove it is a partially ordered precategory.

PROOF. It trivially follows from the definition of partially ordered precategory with star-morphisms. □

17.11. Product of an arbitrary number of funcoids

In this section it will be defined a product of an arbitrary (possibly infinite) indexed family of funcoids.

17.11.1. Mapping a morphism into a pointfree funcoid.

DEFINITION 1332. Let's define the pointfree funcoid χf for every morphism f or a quasi-invertible category:

$$\langle \chi f \rangle a = f \circ a \quad \text{and} \quad \left\langle (\chi f)^{-1} \right\rangle b = f^{\dagger} \circ b.$$

We need to prove it is really a pointfree funcoid.

PROOF. $b \not\asymp \langle \chi f \rangle a \Leftrightarrow b \not\asymp f \circ a \Leftrightarrow a \not\asymp f^{\dagger} \circ b \Leftrightarrow a \not\asymp \left\langle (\chi f)^{-1} \right\rangle b$. □

REMARK 1333. $\langle \chi f \rangle = (f \circ -)$ is the Mor-functor[1] $\mathrm{Mor}(f, -)$ and we can apply Yoneda lemma to it. (See any category theory book for definitions of these terms.)

OBVIOUS 1334. $\langle \chi(g \circ f) \rangle a = g \circ f \circ a$ for composable morphisms f and g or a quasi-invertible category.

17.11.2. General cross-composition product.

DEFINITION 1335. Let fix a quasi-invertible category with with star-morphisms. If f is an indexed family of morphisms from its base category, then the pointfree funcoid $\prod^{(C)} f$ (*cross-composition product* of f) from $\mathrm{StarMor}(\lambda i \in \mathrm{dom}\, f : \mathrm{Src}\, f_i)$ to $\mathrm{StarMor}(\lambda i \in \mathrm{dom}\, f : \mathrm{Dst}\, f_i)$ is defined by the formulas (for all star-morphisms a and b of these forms):

$$\left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(a; f) \quad \text{and} \quad \left\langle \left(\prod^{(C)} f \right)^{-1} \right\rangle b = \mathrm{StarComp}(b; f^{\dagger}).$$

It is really a pointfree funcoid by the definition of quasi-invertible category with star-morphisms.

[1] Also called Hom-functor.