Proposition 1214. If every $\mathfrak{A}_i$ is a poset then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$ whenever every $\bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$ exists;

2°. $\bigsqcap S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcap \left\{ \frac{x_i}{x \in S} \right\}$ whenever every $\bigsqcap \left\{ \frac{x_i}{x \in S} \right\}$ exists.

Proof. It's enough to prove the first formula.

$\left(\lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup \left\{ \frac{x_i}{x \in S} \right\}\right)_i = \bigsqcup \left\{ \frac{x_i}{x \in S} \right\} \sqsupseteq x_i$ for every $x \in S$ and $i \in \operatorname{dom} \mathfrak{A}$.

Let $y \sqsupseteq x$ for every $x \in S$. Then $y_i \sqsupseteq x_i$ for every $i \in \operatorname{dom} \mathfrak{A}$ and thus $y_i \sqsupseteq \bigsqcup \left\{ \frac{x_i}{x \in S} \right\} = \left(\lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup \left\{ \frac{x_i}{x \in S} \right\}\right)_i$ that is $y \sqsupseteq \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$.

Thus $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$ by the definition of join. □

Corollary 1215. If $\mathfrak{A}_i$ are posets then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$ whenever $\bigsqcup S$ exists;

2°. $\bigsqcap S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcap \left\{ \frac{x_i}{x \in S} \right\}$ whenever $\bigsqcap S$ exists.

Proof. It is enough to prove that (for every i) $\bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$ exists whenever $\bigsqcup S$ exists.

Fix $i \in \operatorname{dom} \mathfrak{A}$.

Take $y_i = \left(\bigsqcup S\right)_i$ and let prove that y_i is the least upper bound of $\bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$.

y_i is an upper bound of $\bigsqcup \left\{ \frac{x_i}{x \in S} \right\}$ because $\bigsqcup S \sqsupseteq x$ and thus $\left(\bigsqcup S\right)_i \sqsupseteq x_i$ for every $x \in S$.

Let $x \in S$ and for some $t \in \mathfrak{A}_i$

$$T(t) = \lambda j \in \operatorname{dom} \mathfrak{A} : \begin{cases} t & \text{if } i = j \\ x_i & \text{if } i \neq j. \end{cases}$$

Let $t \sqsupseteq x_i$. Then $T(t) \sqsupseteq x$ for every $x \in S$. So $T(t) \sqsupseteq \bigsqcup S$ and consequently $t = T(t)_i \sqsupseteq y_i$.

So y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$. □

Corollary 1216. If $\mathfrak{A}_i$ are complete lattices then $\mathfrak{A}$ is a complete lattice.

Obvious 1217. If $\mathfrak{A}_i$ are complete (co-)brouwerian lattices then $\mathfrak{A}$ is a (co-)brouwerian lattice.

Proposition 1218. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

Proof. Let $a \neq b$. Then $\exists i \in \operatorname{dom} \mathfrak{A} : a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = (((\operatorname{dom} \mathfrak{A}) \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then $y \not\asymp a$ and $y \asymp b$. □

Obvious 1219. If every $\mathfrak{A}_i$ is a poset with least element $\bot_i$, then the set of atoms of $\prod \mathfrak{A}$ is

$$\left\{ \frac{(\{k\} \times \operatorname{atoms}^{\mathfrak{A}_k}) \cup (\lambda i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\} : \bot_i)}{k \in \operatorname{dom} \mathfrak{A}} \right\}.$$

Proposition 1220. If every $\mathfrak{A}_i$ is an atomistic poset with least element $\bot_i$, then $\prod \mathfrak{A}$ is an atomistic poset.

Proof. $x_i = \bigsqcup \operatorname{atoms} x_i$ for every $x_i \in \mathfrak{A}_i$. Thus

$$x = \lambda i \in \operatorname{dom} x : x_i = \lambda i \in \operatorname{dom} x : \bigsqcup \operatorname{atoms} x_i = \bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x : \begin{cases} x_i & \text{if } j = i \\ \bot_i & \text{if } j \neq i. \end{cases}$$

Take join two times. □

Corollary 1221. If $\mathfrak{A}_i$ are atomistic posets with least elements, then $\prod \mathfrak{A}$ is atomically separable.