17.2. Function spaces of posets

DEFINITION 1206. Let $\mathfrak{A}_i$ be a family of posets indexed by some set $\operatorname{dom}\mathfrak{A}$. We will define order of families of posets by the formula

$$a \sqsubseteq b \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A} : a_i \sqsubseteq b_i.$$

I will call this new poset $\mathfrak{A} = \prod \mathfrak{A}$ *the function space* of posets and the above order *product order*.

PROPOSITION 1207. The function space for posets is also a poset.

PROOF.

Reflexivity. Obvious.
Antisymmetry. Obvious.
Transitivity. Obvious.

□

OBVIOUS 1208. $\mathfrak{A}$ has least element iff each $\mathfrak{A}_i$ has a least element. In this case

$$\bot^{\mathfrak{A}} = \prod_{i \in \operatorname{dom}\mathfrak{A}} \bot^{\mathfrak{A}_i}.$$

PROPOSITION 1209. $a \not\asymp b \Leftrightarrow \exists i \in \operatorname{dom}\mathfrak{A} : a_i \not\asymp b_i$ for every $a, b \in \prod \mathfrak{A}$ if every $\mathfrak{A}_i$ has least element.

PROOF. If $\operatorname{dom}\mathfrak{A} = \emptyset$, then $a = b = \bot$, $a \asymp b$ and thus the theorem statement holds. Assume $\operatorname{dom}\mathfrak{A} \neq \emptyset$.FiXme: Is considering this special case necessary?

$$\begin{aligned} & a \not\asymp b \Leftrightarrow \\ \exists c \in \prod \mathfrak{A} \setminus \{\bot^{\prod \mathfrak{A}}\} &: (c \sqsubseteq a \wedge c \sqsubseteq b) \Leftrightarrow \\ \exists c \in \prod \mathfrak{A} \setminus \{\bot^{\prod \mathfrak{A}}\} \forall i \in \operatorname{dom}\mathfrak{A} &: (c_i \sqsubseteq a_i \wedge c_i \sqsubseteq b_i) \Leftrightarrow \\ \text{(for the reverse implication take } c_j &= \bot^{\mathfrak{A}_j} \text{ for } i \neq j) \\ \exists i \in \operatorname{dom}\mathfrak{A}, c \in \mathfrak{A}_i \setminus \{\bot^{\mathfrak{A}_i}\} &: (c_i \sqsubseteq a_i \wedge c_i \sqsubseteq b_i) \Leftrightarrow \\ \exists i \in \operatorname{dom}\mathfrak{A} &: a_i \not\asymp b_i. \end{aligned}$$

□

PROPOSITION 1210.

1°. If $\mathfrak{A}_i$ are join-semilattices then $\mathfrak{A}$ is a join-semilattice and

$$A \sqcup B = \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi. \tag{25}$$

2°. If $\mathfrak{A}_i$ are meet-semilattices then $\mathfrak{A}$ is a meet-semilattice and

$$A \sqcap B = \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcap Bi.$$

PROOF. It is enough to prove the formula (25).

It's obvious that $\lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi \sqsupseteq A, B$.

Let $C \sqsupseteq A, B$. Then (for every $i \in \operatorname{dom}\mathfrak{A}$) $Ci \sqsupseteq Ai$ and $Ci \sqsupseteq Bi$. Thus $Ci \sqsupseteq Ai \sqcup Bi$ that is $C \sqsupseteq \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi$. □

COROLLARY 1211. If $\mathfrak{A}_i$ are lattices then $\mathfrak{A}$ is a lattice.

OBVIOUS 1212. If $\mathfrak{A}_i$ are distributive lattices then $\mathfrak{A}$ is a distributive lattice.

PROPOSITION 1213. If $\mathfrak{A}_i$ are boolean lattices then $\prod \mathfrak{A}$ is a boolean lattice.

PROOF. We need to prove only that every element $a \in \prod \mathfrak{A}$ has a complement. But this complement is evidently $\lambda i \in \operatorname{dom} a : \overline{a_i}$. □