PROPOSITION 1201. $a \left[f \times^{(C)} g\right] b \Leftrightarrow a \circ f^\dagger \not\asymp g^\dagger \circ b$.

PROOF. From the definition. $\square$

PROPOSITION 1202. $a \left[f \times^{(C)} g\right] b \Leftrightarrow f \left[a \times^{(C)} b\right] g$.

PROOF. $f \left[a \times^{(C)} b\right] g \Leftrightarrow f \circ a^\dagger \not\asymp b^\dagger \circ g \Leftrightarrow a \circ f^\dagger \not\asymp g^\dagger \circ b \Leftrightarrow a \left[f \times^{(C)} g\right] b$. $\square$

THEOREM 1203. $(f \times^{(C)} g)^{-1} = f^\dagger \times^{(C)} g^\dagger$.

PROOF. For every funcoids $a \in \operatorname{Mor}(\operatorname{Src} f; \operatorname{Src} g)$ and $b \in \operatorname{Mor}(\operatorname{Dst} f; \operatorname{Dst} g)$ we have:
$\langle (f \times^{(C)} g)^{-1} \rangle b = g^\dagger \circ b \circ f = \langle f^\dagger \times^{(C)} g^\dagger \rangle b$. $\square$

$\langle ((f \times^{(C)} g)^{-1})^{-1} \rangle a = \langle f \times^{(C)} g \rangle a = g \circ a \circ f^\dagger = \langle (f^\dagger \times^{(C)} g^\dagger)^{-1} \rangle a$.

THEOREM 1204. Let f, g be pointfree funcoids between filters on boolean lattices. Then for every filters $\mathcal{A}_0 \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B}_0 \in \mathfrak{F}(\operatorname{Src} g)$

$$\left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0.$$

PROOF. FiXme: No reason to restrict to atomic filters?For every atom $a_1 \times^{\mathsf{FCD}} b_1$ ($a_1 \in \operatorname{atoms}^{\operatorname{Dst} f}$, $b_1 \in \operatorname{atoms}^{Dstg}$) (see theorem 1132) of the lattice of funcoids we have:

$$\begin{aligned} a_1 \times^{\mathsf{FCD}} b_1 \not\asymp \left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \Leftrightarrow \\ \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g\right] a_1 \times^{\mathsf{FCD}} b_1 \Leftrightarrow \\ (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \circ f^{-1} \not\asymp g^{-1} \circ (a_1 \times^{\mathsf{FCD}} b_1) \Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} \langle g^{-1} \rangle b_1 \Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \not\asymp a_1 \wedge \langle g^{-1} \rangle b_1 \not\asymp \mathcal{B}_0 \Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \not\asymp a_1 \wedge \langle g \rangle \mathcal{B}_0 \not\asymp b_1 \Leftrightarrow \\ \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} b_1. \end{aligned}$$

Thus $\left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0$ because the lattice $\mathsf{FCD}(\mathfrak{F}(\operatorname{Dst} f); \mathfrak{F}(\operatorname{Dst} g))$ is atomically separable (corollary 653). $\square$

COROLLARY 1205. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g\right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$ for every $\mathcal{A}_0 \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{A}_1 \in \mathfrak{F}(\operatorname{Dst} f)$, $\mathcal{B}_0 \in \mathfrak{F}(\operatorname{Src} g)$, $\mathcal{B}_1 \in \mathfrak{F}(\operatorname{Dst} g)$ where $\operatorname{Src} f$, $\operatorname{Dst} f$, $\operatorname{Src} g$, $\operatorname{Dst} g$ are boolean lattices.

PROOF.

$$\begin{aligned} \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g\right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \\ \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \left\langle f \times^{(C)} g \right\rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \Leftrightarrow \\ \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0 \Leftrightarrow \\ \mathcal{A}_1 \not\asymp \langle f \rangle \mathcal{A}_0 \wedge \mathcal{B}_1 \not\asymp \langle f \rangle \mathcal{B}_0 \Leftrightarrow \\ \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1. \end{aligned}$$

$\square$