CHAPTER 17

Multifuncoids and staroids

17.1. Product of two funcoids

17.1.1. Lemmas. FiXme: Previous theorems/propositions shall not be repeated as lemmas.

LEMMA 1196. Let A, B, C be sets, $f \in \mathsf{FCD}(A; B)$, $g \in \mathsf{FCD}(B; C)$, $h \in \mathsf{FCD}(A; C)$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF. Proposition 637. □

LEMMA 1197. Let A, B, C be sets, $f \in \mathsf{RLD}(A; B)$, $g \in \mathsf{RLD}(B; C)$, $h \in \mathsf{RLD}(A; C)$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF. Theorem 746. □

17.1.2. Definition.

DEFINITION 1198. I will call a *quasi-invertible category* a partially ordered dagger category such that it holds

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{\dagger} \tag{24}$$

for every morphisms $f \in \mathrm{Mor}(A; B)$, $g \in \mathrm{Mor}(B; C)$, $h \in \mathrm{Mor}(A; C)$, where A, B, C are objects of this category.

Inverting this formula, we get $f^{\dagger} \circ g^{\dagger} \not\asymp h^{\dagger} \Leftrightarrow g^{\dagger} \not\asymp f \circ h^{\dagger}$. After replacement of variables, this gives: $f^{\dagger} \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$.

As it follows from above, the category of funcoids and the category of reloids are quasi-invertible (taking $f^{\dagger} = f^{-1}$). Moreover the category of pointfree funcoids between lattices of filters on boolean lattices is quasi-invertible (theorem 1119).FiXme: Say that **Rel** is quasi-invertible.

EXERCISE 1199. Prove that every ordered groupoid is quasi-invertible category if we define the dagger as the inverse morphism.

DEFINITION 1200. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree funcoid $\mathrm{Mor}(\mathrm{Src}\, f; \mathrm{Src}\, g) \to \mathrm{Mor}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$ defined by the formulas (for every $a \in \mathrm{Mor}(\mathrm{Src}\, f; \mathrm{Src}\, g)$ and $b \in \mathrm{Mor}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$):

$$\left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^{\dagger} \quad \text{and} \quad \left\langle (f \times^{(C)} g)^{-1} \right\rangle b = g^{\dagger} \circ b \circ f.$$

We need to prove that it is really a pointfree funcoid that is that

$$b \not\asymp \left\langle f \times^{(C)} g \right\rangle a \Leftrightarrow a \not\asymp \left\langle (f \times^{(C)} g)^{-1} \right\rangle b.$$

This formula means $b \not\asymp g \circ a \circ f^{\dagger} \Leftrightarrow a \not\asymp g^{\dagger} \circ b \circ f$ and can be easily proved applying the formula (24) two times.