every $a, b \in \operatorname{atoms}^{\mathfrak{B}}$. This is possible only (corollary 1112 and the fact that $\mathfrak{B}$ is atomic) when $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}$.

$\neg 2^\circ \Rightarrow \neg 1^\circ$. Suppose $\langle f\rangle a \notin \operatorname{atoms}^{\mathfrak{B}} \cup \{\bot^{\mathfrak{B}}\}$ for some $a \in \operatorname{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p \neq q$ such that $\langle f\rangle a \sqsupseteq p \wedge \langle f\rangle a \sqsupseteq q$. Consequently $p \sqcap \langle f\rangle a \neq 0^{\mathfrak{B}}$; $a \sqcap \langle f^{-1}\rangle p \neq \bot^{\mathfrak{A}}$; $a \sqsubseteq \langle f^{-1}\rangle p$; $\langle f \circ f^{-1}\rangle p = \langle f\rangle\langle f^{-1}\rangle p \sqsupseteq \langle f\rangle a \sqsupseteq q$ (by proposition 1070 because $\mathfrak{B}$ is separable by proposition 181); $\langle f \circ f^{-1}\rangle p \not\sqsubseteq p$ and $\langle f \circ f^{-1}\rangle p \neq \bot^{\mathfrak{B}}$. So it cannot be $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}$. □

THEOREM 1161. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ be primary filtrators over a boolean lattice. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is monovalued iff

$$\forall I, J \in \mathfrak{Z}_1 : \langle f^{-1}\rangle(I \sqcap^{\mathfrak{Z}_1} J) = \langle f^{-1}\rangle I \sqcap \langle f^{-1}\rangle J.$$

PROOF. $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices (corollary 374).

$(\mathfrak{B}; \mathfrak{Z}_1)$ is a filtrator with separable core by the theorem 379.

$(\mathfrak{B}; \mathfrak{Z}_1)$ is finitely meet-closed by the theorem 364.

$\mathfrak{A}$ and $\mathfrak{B}$ are starrish by corollary 381.

$\mathfrak{A}$ is separable by obvious 403.

We are under conditions of the theorem 1081.

$\Rightarrow$. Obvious (taking into account that $(\mathfrak{B}; \mathfrak{Z}_1)$ is finitely meet-closed).

$\Leftarrow$.

$$\begin{aligned}
\langle f^{-1}\rangle(\mathcal{I} \sqcap \mathcal{J}) &= \\
\bigsqcap \langle\langle f^{-1}\rangle\rangle^* \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)}(\mathcal{I} \sqcap \mathcal{J}) &= \\
\bigsqcap \langle\langle f^{-1}\rangle\rangle^* \left\{ \frac{I \sqcap^{\mathfrak{Z}_1} J}{I \in \operatorname{up}\mathcal{I}, J \in \operatorname{up}\mathcal{J}} \right\} &= \\
\bigsqcap \left\{ \frac{\langle f^{-1}\rangle(I \sqcap^{\mathfrak{Z}_1} J)}{I \in \operatorname{up}\mathcal{I}, J \in \operatorname{up}\mathcal{J}} \right\} &= \\
\bigsqcap \left\{ \frac{\langle f^{-1}\rangle I \sqcap \langle f^{-1}\rangle J}{I \in \operatorname{up}\mathcal{I}, J \in \operatorname{up}\mathcal{J}} \right\} &= \\
\bigsqcap \left\{ \frac{\langle f^{-1}\rangle I}{I \in \operatorname{up}\mathcal{I}} \right\} \sqcap \bigsqcap \left\{ \frac{\langle f^{-1}\rangle J}{J \in \operatorname{up}\mathcal{J}} \right\} &= \\
\langle f^{-1}\rangle\mathcal{I} \sqcap^{\mathfrak{A}} \langle f^{-1}\rangle\mathcal{J}
\end{aligned}$$

(used theorem 1081, theorem 377, theorem 1071).

□

15.14. Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset. Let $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{A})$.

DEFINITION 1162. Let's call *closed* regarding a pointfree funcoid f such element $a \in \mathfrak{A}$ that $\langle f\rangle a \sqsubseteq a$.

PROPOSITION 1163. If i and j are closed (regarding a pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{A})$), S is a set of closed elements (regarding f), then

1°. $i \sqcup j$ is a closed element, if $\mathfrak{A}$ is a separable starrish join-semilattice;

2°. $\bigsqcap S$ is a closed element if $\mathfrak{A}$ is a separable complete lattice.

PROOF. $\langle f\rangle(i \sqcup j) = \langle f\rangle i \sqcup \langle f\rangle j \sqsubseteq i \sqcup j$ (theorem 1071), $\langle f\rangle \bigsqcap S \sqsubseteq \bigsqcap \langle\langle f\rangle\rangle^* S \sqsubseteq \bigsqcap S$ (used separability of $\mathfrak{A}$ twice). Consequently the elements $i \sqcup j$ and $\bigsqcap S$ are closed. □