Let S be a set of filters.

$$\begin{aligned}\langle f\rangle\bigsqcup S &=\\ \bigsqcup\langle\langle f\rangle\rangle^*\operatorname{atoms}\bigsqcup S &=\\ \bigsqcup\langle\langle f\rangle\rangle^*\bigcup\langle\operatorname{atoms}\rangle^* S &=\\ \bigsqcup\bigcup\langle\langle\langle f\rangle\rangle^*\rangle^*\langle\operatorname{atoms}\rangle^* S &=\\ \bigsqcup\left\langle\bigsqcup\right\rangle^*\langle\langle\langle f\rangle\rangle^*\rangle^*\langle\operatorname{atoms}\rangle^* S &=\\ \bigsqcup\left\langle\bigsqcup\circ\langle\langle f\rangle\rangle^*\circ\operatorname{atoms}\right\rangle^* S &=\\ \bigsqcup\left\{\frac{\bigsqcup\langle\langle f\rangle\rangle^*\operatorname{atoms} a}{a\in S}\right\} &=\\ \bigsqcup\left\{\frac{\langle f\rangle a}{a\in S}\right\} &=\\ \bigsqcup\langle\langle f\rangle\rangle^* S.\end{aligned}$$

□

Definition 1145. Let $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ be join-semilattices with least elements. I will call *pointfree generalized closure* such a function $\alpha\in(\mathfrak{Z}_1)^{\mathfrak{Z}_0}$ thatFiXme: It is just a map preserving finite joins, no need to introduce a new term. It can be generalized for arbitrary posets.

1°. $\alpha\bot^{\mathfrak{Z}_0}=\bot^{\mathfrak{Z}_1}$;
2°. $\forall I,J\in\mathfrak{Z}_0:\alpha(I\sqcup J)=\alpha I\sqcup\alpha J$.

Definition 1146. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ be primary filtrators over boolean lattices. I will call a *co-complete pointfree funcoid* a pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ such that $\langle f\rangle|_{\mathfrak{Z}_0}$ is a pointfree generalized closure.

Proposition 1147. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Co-complete pointfree funcoids $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ bijectively correspond to pointfree generalized closures $\mathfrak{Z}_1^{\mathfrak{Z}_0}$, where the bijection is $f\mapsto\langle f\rangle|_{\mathfrak{Z}_0}$.

Proof. It follows from the theorem 1082. □

Theorem 1148. Let $(\mathfrak{A};\mathfrak{Z}_0)$ be semifiltered, star-separable, down-aligned filtrator with finitely meet closed, join-closed, and separable core, where $\mathfrak{Z}_0$ is a complete boolean lattice and both $\mathfrak{Z}_0$ and $\mathfrak{A}$ are atomistic lattices.

Let $(\mathfrak{B};\mathfrak{Z}_1)$ be a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1°. f^{-1} is co-complete;
2°. $\forall S\in\mathscr{P}\mathfrak{A},J\in\mathfrak{Z}_1:(\bigsqcup^{\mathfrak{A}}S\ [f]\ J\Rightarrow\exists\mathcal{I}\in S:\mathcal{I}\ [f]\ J)$;
3°. $\forall S\in\mathscr{P}\mathfrak{Z}_0,J\in\mathfrak{Z}_1:(\bigsqcup^{\mathfrak{Z}_0}S\ [f]\ J\Rightarrow\exists I\in S:I\ [f]\ J)$;
4°. f is complete;
5°. $\forall S\in\mathscr{P}\mathfrak{Z}_0:\langle f\rangle\bigsqcup^{\mathfrak{Z}_0}S=\bigsqcup^{\mathfrak{B}}\langle\langle f\rangle\rangle^*S$.

Proof. First note that the theorem 320 applies to the filtrator $(\mathfrak{A};\mathfrak{Z}_0)$.

3°⇒1°. For every $S\in\mathscr{P}\mathfrak{Z}_0$, $J\in\mathfrak{Z}_1$

$$\bigsqcup^{\mathfrak{Z}_0}S\sqcap^{\mathfrak{A}}\langle f^{-1}\rangle J\neq\bot^{\mathfrak{A}}\Rightarrow\exists I\in S:I\sqcap^{\mathfrak{A}}\langle f^{-1}\rangle J\neq\bot^{\mathfrak{A}},\tag{22}$$

consequently by the theorem 320 we have $\langle f^{-1}\rangle J\in\mathfrak{Z}_0$.