COROLLARY 1140. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a co-brouwerian lattice.

PROPOSITION 1141. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be sets of filters over some boolean lattices and $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B};\mathfrak{C})$. Let $\mathfrak{B}$ be an atomic poset. Then

$$\operatorname{atoms}(g \circ f) = \left\{ \frac{x \times^{\mathsf{FCD}} z}{x \in \operatorname{atoms}^{\mathfrak{A}}, z \in \operatorname{atoms}^{\mathfrak{C}}, \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g)} \right\}.$$

PROOF.

$$\begin{aligned} (x \times^{\mathsf{FCD}} z) \sqcap (g \circ f) \neq \bot^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})} &\Leftrightarrow \\ x \, [g \circ f] \, z &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \, [f] \, y \wedge y \, [g] \, z) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((x \times^{\mathsf{FCD}} y) \sqcap f \neq \bot^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \wedge (y \times^{\mathsf{FCD}} z) \sqcap g \neq \bot^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})}) & \end{aligned}$$

(were used corollary 1125 and theorem 1117). □

THEOREM 1142. Let f be a pointfree funcoid between sets of filters on boolean lattices.

1°. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$;
2°. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

PROOF.

1°.

$$\begin{aligned} \exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y} &\Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge \mathcal{X} \left[a \times^{\mathsf{FCD}} b\right] \mathcal{Y}) &\Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge a \times^{\mathsf{FCD}} b \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) &\Leftrightarrow \\ \exists F \in \operatorname{atoms} f : (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) &\Leftrightarrow \\ f \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} &\Leftrightarrow \\ \mathcal{X} \, [f] \, \mathcal{Y}. & \end{aligned}$$

2°. Let $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \, [f] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious.

□

15.11. Complete pointfree funcoids

DEFINITION 1143. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is *complete*, when for every $S \in \mathscr{P}\mathfrak{A}$ whenever both $\bigsqcup S$ and $\bigsqcup \langle \langle f \rangle \rangle^* S$ are defined we have

$$\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S.$$

PROPOSITION 1144. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is complete iff $\langle f \rangle a = \bigsqcup \langle \langle f \rangle \rangle^* \operatorname{atoms} a$ for every $a \in \mathfrak{A}$.

PROOF. Direct implication is obvious. The reverse implication: