PROOF. I will prove only the first equality because the other is analogous.

We can apply theorem 1091.

For every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{C}$

$$\begin{aligned} &\mathcal{X} \left[ f \circ (g \sqcup h) \right] \mathcal{Z} \Leftrightarrow \\ &\exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \left[ g \sqcup h \right] y \wedge y \left[ f \right] \mathcal{Z}) \Leftrightarrow \\ &\exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((\mathcal{X} \left[ g \right] y \vee \mathcal{X} \left[ h \right] y) \wedge y \left[ f \right] \mathcal{Z}) \Leftrightarrow \\ &\exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((\mathcal{X} \left[ g \right] y \wedge y \left[ f \right] \mathcal{Z}) \vee (\mathcal{X} \left[ h \right] y \wedge y \left[ f \right] \mathcal{Z})) \Leftrightarrow \\ &\exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \left[ g \right] y \wedge y \left[ f \right] \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \left[ h \right] y \wedge y \left[ f \right] \mathcal{Z}) \Leftrightarrow \\ &\mathcal{X} \left[ f \circ g \right] \mathcal{Z} \vee \mathcal{X} \left[ f \circ h \right] \mathcal{Z} \Leftrightarrow \\ &\mathcal{X} \left[ f \circ g \sqcup f \circ h \right] \mathcal{Z}. \end{aligned}$$

Thus $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ by theorem 1068. □

THEOREM 1119. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be posets of filters over some boolean lattices, $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$, $h \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{C})$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF.

$$\begin{aligned} &g \circ f \not\asymp h \Leftrightarrow \\ &\exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : a \left[ (g \circ f) \sqcap h \right] c \Leftrightarrow \\ &\exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (a \left[ g \circ f \right] c \wedge a \left[ h \right] c) \Leftrightarrow \\ &\exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (a \left[ f \right] b \wedge b \left[ g \right] c \wedge a \left[ h \right] c) \Leftrightarrow \\ &\exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (b \left[ g \right] c \wedge b \left[ h \circ f^{-1} \right] c) \Leftrightarrow \\ &\exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : b \left[ g \sqcap (h \circ f^{-1}) \right] c \Leftrightarrow \\ &g \not\asymp h \circ f^{-1}. \end{aligned}$$

□

## 15.9. Funcoidal product of elements

DEFINITION 1120. *Funcoidal product* $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ where $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements is a pointfree funcoid such that for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A}; \\ \bot^{\mathfrak{B}} & \text{if } \mathcal{X} \asymp \mathcal{A}; \end{cases} \quad \text{and} \quad \langle (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})^{-1} \rangle \mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \not\asymp \mathcal{B}; \\ \bot^{\mathfrak{A}} & \text{if } \mathcal{Y} \asymp \mathcal{B}. \end{cases}$$

PROPOSITION 1121. $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a pointfree funcoid and

$$\mathcal{X} \left[ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B}.$$

PROOF. Obvious. □

PROPOSITION 1122. Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable bounded posetsFiXme: Switch to correct conditions for existence and due properties of image and domain., $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. ThenFiXme: Should add proposition with one-direction implication, without separability requirement.

$$f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}.$$

PROOF. If $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \sqsubseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$, $\operatorname{im} f \sqsubseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{B}$. If $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$ then $\mathcal{X} \left[ f \right] \mathcal{Y} \Rightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Rightarrow \mathcal{Y} \not\asymp \mathcal{B}$ and similarly $\mathcal{X} \left[ f \right] \mathcal{Y} \Rightarrow \mathcal{X} \not\asymp \mathcal{A}$.

So $\left[ f \right] \subseteq \left[ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right]$ and thus using separability $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □