2°. $\operatorname{dom}(g \circ f) = \operatorname{im}(f^{-1} \circ g^{-1})$ what by the proved is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$.

□

15.6. Category of pointfree funcoids

I will define the category pFCD of pointfree funcoids:

- The class of objects are small posets.
- The set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object $\mathfrak{A}$ is $(\mathfrak{A}; \mathfrak{A}; \mathrm{id}_{\mathfrak{A}}; \mathrm{id}_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

The *category of pointfree funcoid triples*FiXme: They are quintuples not triples. is defined as follows:

- Objects are pairs $(\mathfrak{A}; \mathcal{A})$ where $\mathfrak{A}$ is a small poset and $\mathcal{A} \in \mathfrak{A}$.
- The morphisms from an object $(\mathfrak{A}; \mathcal{A})$ to an object $(\mathfrak{B}; \mathcal{B})$ are tuples $(\mathfrak{A}; \mathfrak{B}; \mathcal{A}; \mathcal{B}; f)$ where $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$.FiXme: Domain and image are not always defined. Even if it's defined, the composition law may not hold. We can require instead $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$, but this is defined only for posets with least elements.
- The composition is defined by the formula $(\mathcal{B}; \mathcal{C}; g) \circ (\mathcal{A}; \mathcal{B}; f) = (\mathcal{A}; \mathcal{C}; g \circ f)$.
- Identity morphism for an object $(\mathfrak{A}; \mathcal{A})$ is $\mathrm{id}^{\mathsf{FCD}(\mathfrak{A})}_{\mathcal{A}}$.FiXme: Defined only for meet-semilattices. We can also define a wider precategory without identity.

To prove that it is really a category is trivial.

15.7. Specifying funcoids by functions or relations on atomic filters

THEOREM 1110. Let $\mathfrak{A}$ be an atomic poset and $(\mathfrak{B}; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}.$$

PROOF. For every $Y \in \mathfrak{Z}_1$ we have

$$Y \not\asymp^{\mathfrak{B}} \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : Y \not\asymp^{\mathfrak{B}} \langle f \rangle x.$$

Thus $\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle^* \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} = \partial \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ (used theorem 399). Consequently $\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 395. □

PROPOSITION 1111. Let f be a pointfree funcoid. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

1°. $\mathcal{X} \,[f]\, \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X} : x \,[f]\, \mathcal{Y}$ if $\operatorname{Src} f$ is an atomic poset.
2°. $\mathcal{X} \,[f]\, \mathcal{Y} \Leftrightarrow \exists y \in \operatorname{atoms} \mathcal{Y} : \mathcal{X} \,[f]\, y$ if $\operatorname{Dst} f$ is an atomic poset.

PROOF. I will prove only the second as the first is similar.

If $\mathcal{X} \,[f]\, \mathcal{Y}$, then $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$, consequently exists $y \in \operatorname{atoms} \mathcal{Y}$ such that $y \not\asymp \langle f \rangle \mathcal{X}$, $\mathcal{X} \,[f]\, y$. The reverse is obvious. □

COROLLARY 1112. If f is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

$$\mathcal{X} \,[f]\, \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \,[f]\, y.$$