PROPOSITION 1078. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable pointfree funcoids f, g, h.

PROOF. $\langle (h \circ g) \circ f \rangle = \langle h \circ g \rangle \circ \langle f \rangle = \langle h \rangle \circ \langle g \rangle \circ \langle f \rangle = \langle h \rangle \circ \langle g \circ f \rangle = \langle h \circ (g \circ f) \rangle$;

$$\left\langle ((h \circ g) \circ f)^{-1} \right\rangle = \left\langle f^{-1} \circ (h \circ g)^{-1} \right\rangle = \left\langle f^{-1} \circ g^{-1} \circ h^{-1} \right\rangle = \left\langle (g \circ f)^{-1} \circ h^{-1} \right\rangle = \left\langle (h \circ (g \circ f))^{-1} \right\rangle.$$

□

15.3. Pointfree funcoid as continuation

PROPOSITION 1079. Let f be a pointfree funcoid. Then for every $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$ we have

1°. If $(\operatorname{Src} f; \mathfrak{Z})$ is a filtrator with separable core then $x \; [f] \; y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z})} x : X \; [f] \; y$.

2°. If $(\operatorname{Dst} f; \mathfrak{Z})$ is a filtrator with separable core then $x \; [f] \; y \Leftrightarrow \forall Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z})} y : x \; [f] \; Y$.

PROOF. We will prove only the second because the first is similar.

$x \; [f] \; y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z})} y : Y \not\asymp \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z})} y : x \; [f] \; Y$.

□

COROLLARY 1080. Let f be a pointfree funcoid and $(\operatorname{Src} f; \mathfrak{Z}_0)$, $(\operatorname{Dst} f; \mathfrak{Z}_1)$ are filtrators with separable core. Then

$$x \; [f] \; y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x, Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z}_1)} y : X \; [f] \; Y.$$

PROOF. Apply the proposition twice. □

THEOREM 1081. Let f be a pointfree funcoid. Let $(\operatorname{Src} f; \mathfrak{Z}_0)$ be a finitely meet-closed filtrator with separable core which is a meet-semilattice and $\forall x \in \operatorname{Src} f : \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x \neq \emptyset$ and $(\operatorname{Dst} f; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice.

$$\langle f \rangle x = \prod^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x.$$

PROOF. By the previous proposition for every $y \in \operatorname{Dst} f$:

$y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow x \; [f] \; y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x : X \; [f] \; y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x : y \not\asymp^{\operatorname{Dst} f} \langle f \rangle X$.

Let's denote $W = \left\{ \frac{y \sqcap^{\operatorname{Dst} f} \langle f \rangle^* X}{X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x} \right\}$. We will prove that W is a generalized filter base over $\mathfrak{Z}_1$. To prove this enough to show that $V = \left\{ \frac{\langle f \rangle^* X}{X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f \rangle^* A$, $\mathcal{Q} = \langle f \rangle^* B$ where $A, B \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x$; $A \sqcap^{\mathfrak{Z}_0} B \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x$ (used the fact that it is a finitely meet-closed and theorem 311) and $\mathcal{R} \sqsubseteq \mathcal{P} \sqcap^{\operatorname{Dst} f} \mathcal{Q}$ for $\mathcal{R} = \langle f \rangle (A \sqcap^{\mathfrak{Z}_0} B) \in V$ because $\operatorname{Dst} f$ is separable by obvious 403. So V is a generalized filter base and thus W is a generalized filter base.

$\bot^{\operatorname{Dst} f} \notin W \Leftrightarrow \bot^{\operatorname{Dst} f} \notin \prod^{\operatorname{Dst} f} W$ by theorem 388. That is

$$\forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x : y \sqcap^{\operatorname{Dst} f} \langle f \rangle^* X \neq \bot^{\operatorname{Dst} f} \Leftrightarrow y \sqcap^{\operatorname{Dst} f} \prod^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x \neq \bot^{\operatorname{Dst} f}.$$

Comparing with the above,

$$y \sqcap^{\operatorname{Dst} f} \langle f \rangle x \neq \bot^{\operatorname{Dst} f} \Leftrightarrow y \sqcap^{\operatorname{Dst} f} \prod^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x \neq \bot^{\operatorname{Dst} f}.$$