

## Pointfree functors

This chapter is based on [28].

This is a routine chapter. There is almost nothing creative here. I just generalize theorems about functors to the maximum extent for *pointfree functors* (defined below) preserving the proof idea. The main idea behind this chapter is to find the weakest theorem conditions enough for the same theorem statement as for above theorems for functors.

For those who know pointfree topology: Pointfree topology notions of frames and locales is a non-trivial generalization of topological spaces. Pointfree functors are different: I just replace the set of filters on a set with an arbitrary poset, this readily gives the definition of *pointfree functor*, almost no need of creativity here.

Pointfree functors are used in the below definitions of products of functors.

### 15.1. Definition

DEFINITION 1057. *Pointfree functor* is a quadruple  $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$  where  $\mathfrak{A}$  and  $\mathfrak{B}$  are posets,  $\alpha \in \mathfrak{B}^{\mathfrak{A}}$  and  $\beta \in \mathfrak{A}^{\mathfrak{B}}$  such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (y \not\prec \alpha x \Leftrightarrow x \not\prec \beta y).$$

DEFINITION 1058. The *source*  $\text{Src}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{A}$  and *destination*  $\text{Dst}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{B}$  for every pointfree functor  $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ .

To every functor  $(A; B; \alpha; \beta)$  corresponds pointfree functor  $(\mathcal{P}A; \mathcal{P}B; \alpha; \beta)$ . Thus pointfree functors are a generalization of functors.

DEFINITION 1059. I will denote  $\text{FCD}(\mathfrak{A}; \mathfrak{B})$  the set of pointfree functors from  $\mathfrak{A}$  to  $\mathfrak{B}$  (that is with source  $\mathfrak{A}$  and destination  $\mathfrak{B}$ ), for every posets  $\mathfrak{A}$  and  $\mathfrak{B}$ .

PROPOSITION 1060. If  $\mathfrak{A}$  and  $\mathfrak{B}$  have least elements, then  $\text{FCD}(\mathfrak{A}; \mathfrak{B})$  has least element. **FiXme: Move below where order of pointfree functors is defined.**

PROOF. It is  $(\mathfrak{A}; \mathfrak{B}; \mathfrak{A} \times \{\perp^{\mathfrak{B}}\}; \mathfrak{B} \times \{\perp^{\mathfrak{A}}\})$ . □

DEFINITION 1061.  $\langle (\mathfrak{A}; \mathfrak{B}; \alpha; \beta) \rangle \stackrel{\text{def}}{=} \alpha$  for every pointfree functor  $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ .

DEFINITION 1062.  $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha)$  for every pointfree functor  $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ .

PROPOSITION 1063. If  $f$  is a pointfree functor then  $f^{-1}$  is also a pointfree functor.

PROOF. It follows from symmetry in the definition of pointfree functor. □

OBVIOUS 1064.  $(f^{-1})^{-1} = f$  for every pointfree functor  $f$ .

DEFINITION 1065. The relation  $[f] \in \mathcal{P}(\text{Src } f \times \text{Dst } f)$  is defined by the formula (for every pointfree functor  $f$  and  $x \in \text{Src } f$ ,  $y \in \text{Dst } f$ )

$$x [f] y \stackrel{\text{def}}{=} y \not\prec \langle f \rangle x.$$