CONJECTURE 1050. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \sqsubset \mathcal{A} \ltimes \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

A stronger conjecture:

CONJECTURE 1051. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \sqsubset \mathcal{A} \ltimes \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$. Particularly, is this formula true for $\mathcal{A} = \mathcal{B} = \Delta \sqcap \uparrow^{\mathbb{R}} (0; +\infty)$?

The above conjecture is similar to Fermat Last Theorem as having no value by itself but being somehow challenging to prove it (not expected to be as hard as FLT however).

EXAMPLE 1052. $\mathcal{A} \ltimes \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

PROOF. It's enough to prove $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.
Let $\Delta_+ = \Delta \sqcap \uparrow^{\mathbb{R}} (0; +\infty)$. Let $\mathcal{A} = \mathcal{B} = \Delta_+$.
Let $K = (\leq)|_{\mathbb{R} \times \mathbb{R}}$.
Obviously $K \notin \operatorname{GR}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$.
$\mathcal{A} \ltimes \mathcal{B} \sqsubseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A};\mathcal{B})} K$ and thus $K \in \operatorname{GR}(\mathcal{A} \ltimes \mathcal{B})$ because

$$\uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A};\mathcal{B})} K \sqsupseteq \Delta_+ \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B = \mathcal{A} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B$$

for $B = (0; +\infty)$.
Thus $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

EXAMPLE 1053. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.FiXme: Does it hold for some principal filters $\mathcal{A}$, $\mathcal{B}$?

PROOF. This follows from the above example. □

PROPOSITION 1054. $(\mathcal{A} \ltimes \mathcal{B}) \sqcap (\mathcal{A} \rtimes \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$.

PROOF.

$$\begin{aligned}(\mathcal{A} \ltimes \mathcal{B}) \sqcap (\mathcal{A} \rtimes \mathcal{B}) &\sqsubseteq \\ \bigsqcap \left\{ \frac{\uparrow^{\mathsf{RLD}} f}{f \in \mathbf{Rel}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B})), \uparrow^{\mathsf{FCD}} f \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}} \right\} &= \\ \bigsqcap \left\{ \frac{\uparrow^{\mathsf{RLD}} f}{\uparrow^{\mathsf{FCD}} f \in \operatorname{xyGR}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})} \right\} &= \\ (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) &= \\ \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}&.\end{aligned}$$

To finish the proof we need to show $\mathcal{A} \ltimes \mathcal{B} \sqsupseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ and $\mathcal{A} \rtimes \mathcal{B} \sqsupseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$. By symmetry it's enough to show $\mathcal{A} \ltimes \mathcal{B} \sqsupseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ what is proved above. □

EXAMPLE 1055. $(\mathcal{A} \ltimes \mathcal{B}) \sqcup (\mathcal{A} \rtimes \mathcal{B}) \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

PROOF. (based on [8]) Let $\mathcal{A} = \mathcal{B} = \Omega(\mathbb{N})$. It's enough to prove $(\mathcal{A} \ltimes \mathcal{B}) \sqcup (\mathcal{A} \rtimes \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.
Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ that is $X \in \Omega(\mathbb{N})$, $Y \in \Omega(\mathbb{N})$.
Removing one element x from X produces a set P. Removing one element y from Y produces a set Q. Obviously $P \in \Omega(\mathbb{N})$, $Q \in \Omega(\mathbb{N})$.
Obviously $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \in \operatorname{GR}((\mathcal{A} \ltimes \mathcal{B}) \sqcup (\mathcal{A} \rtimes \mathcal{B}))$.
$(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \not\supseteq X \times Y$ because $(x; y) \in X \times Y$ but $(x; y) \notin (P \times \mathbb{N}) \cup (\mathbb{N} \times Q)$.
Thus $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \notin \operatorname{GR}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ by properties of filter bases. □

EXAMPLE 1056. $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD}) f \neq f$ for some convex reloid f.

PROOF. Let $f = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ where $\mathcal{A}$ and B are from example 1053.
$(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ by proposition 807.
So $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □