1°⇔2°.

$$\mathcal{B} = f * \mathcal{A} \Leftrightarrow \mathcal{B} = \left\{ \frac{C \in \mathscr{P}\operatorname{Base}(\mathcal{B})}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\} \Leftrightarrow$$
$$\forall C \in \mathscr{P}\operatorname{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}).$$

2°⇔3°. Because f is a bijection.
2°⇒5°. For every $C \in \mathcal{B}$ we have $\langle f^{-1} \rangle^* C \in \mathcal{A}$ and thus $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}} \langle \uparrow^{\mathsf{FCD}} f^{-1} \rangle C = \langle f \rangle^* \langle f^{-1} \rangle^* C = C$. Thus $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is onto $\mathcal{B}$.
4°⇒5°. Obvious.
5°⇒4°. We need to prove only that $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.
4°⇒3°. We have $\forall C \in \operatorname{Base}(\mathcal{A}) : ((\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}) C \Leftrightarrow C \in \mathcal{A})$ and consequently $\forall C \in \operatorname{Base}(\mathcal{A}) : (\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A})$.
6°⇔1°. From the last corollary.
1°⇔7°. Obvious.
7°⇔8°. Obvious.

□

Corollary 962. The following are equivalent for every filters $\mathcal{A}$ and $\mathcal{B}$:

1°. $\mathcal{A}$ is directly isomorphic to $\mathcal{B}$.
2°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$
$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}.$$
3°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$
$$\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$
4°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is a bijection from $\mathcal{A}$ to $\mathcal{B}$.
5°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is a function onto $\mathcal{B}$.
6°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$.
7°. There is a bijective morphism $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$.

Proposition 963. $\mathbf{GreFunc}_1$ and $\mathbf{GreFunc}_2$ with function composition are categories.

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of $\mathbf{GreFunc}_1$. Then $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B}$. So
$$\langle \uparrow^{\mathsf{FCD}} (g \circ f) \rangle \mathcal{A} = \langle \uparrow^{\mathsf{FCD}} g \rangle \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A} \sqsupseteq \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B} \sqsupseteq \mathcal{C}.$$
Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\mathrm{id}_{\operatorname{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every filter $\mathcal{A}$.

Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of $\mathbf{GreFunc}_2$. Then $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} = \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B}$. So
$$\langle \uparrow^{\mathsf{FCD}} (g \circ f) \rangle \mathcal{A} = \langle \uparrow^{\mathsf{FCD}} g \rangle \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A} = \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B} = \mathcal{C}.$$
Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_2$. Associativity law is evident. $\mathrm{id}_{\operatorname{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_2$ for every filter $\mathcal{A}$. □

Corollary 964. $\leq_1$ and $\leq_2$ are preorders.

Theorem 965. **FuncBij** is a groupoid.