13.2. Ordering of filters

Below I will define some categories having filters (with possibly different bases) as their objects and some relations having two filters (with possibly different bases) as arguments induced by these categories (defined as existence of a morphism between these two filters).

THEOREM 946. $\operatorname{card} a = \operatorname{card} U$ for every ultrafilter a on U if U is infinite.

PROOF. Let $f(X) = X$ if $X \in a$ and $f(X) = U \setminus X$ if $X \notin a$. Obviously f is a surjection from U to a.

Every $X \in a$ appears as a value of f exactly twice, as $f(X)$ and $f(U \setminus X)$. So $\operatorname{card} a = (\operatorname{card} U)/2 = \operatorname{card} U$. □

COROLLARY 947. Cardinality of every two ultrafilters on a set U is the same.

PROOF. For infinite U it follows from the theorem. For finite case it is obvious. □

DEFINITION 948. $f * \mathcal{A} = \left\{ \frac{C \in \mathscr{P}(\operatorname{Dst} f)}{\langle f^{-1}\rangle^* C \in \mathcal{A}} \right\}$ for every filter $\mathcal{A}$ and a **Set**-morphism f.[1]

Below I'll define some directed multigraphs. By an abuse of notation, I will denote these multigraphs the same as (below defined) categories based on some of these directed multigraphs with added composition of morphisms (of directed multigraphs edges). As such I will call vertices of these multigraphs objects and edges morphisms.

DEFINITION 949. I will denote $\mathbf{GreFunc}_1$ the multigraph whose objects are filters and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are **Set**-morphisms from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \supseteq f * \mathcal{A}$.

DEFINITION 950. I will denote $\mathbf{GreFunc}_2$ the multigraph whose objects are filters and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are **Set**-morphisms from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = f * \mathcal{A}$.

DEFINITION 951. Let $\mathcal{A}$ be a filter on a set X and $\mathcal{B}$ is a filter on a set Y. $\mathcal{A} \geq_1 \mathcal{B}$ iff $\operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A}; \mathcal{B})$ is not empty.

DEFINITION 952. Let $\mathcal{A}$ be a filter on a set X and $\mathcal{B}$ is a filter on a set Y. $\mathcal{A} \geq_2 \mathcal{B}$ iff $\operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$ is not empty.

PROPOSITION 953.

1°. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A}; \mathcal{B})$ iff f is a **Set**-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that
$$C \in \mathcal{B} \Leftarrow \langle f^{-1}\rangle^* C \in \mathcal{A}$$
for every $C \in \mathscr{P} \operatorname{Base}(\mathcal{B})$.

2°. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$ iff f is a **Set**-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that
$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1}\rangle^* C \in \mathcal{A}$$
for every $C \in \mathscr{P} \operatorname{Base}(\mathcal{B})$.

PROOF.

1°.

$$f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A}; \mathcal{B}) \Leftrightarrow \mathcal{B} \supseteq f * \mathcal{A} \Leftrightarrow \forall C \in f * \mathcal{A} : C \in \mathcal{B} \Leftrightarrow \forall C \in \mathscr{P} \operatorname{Base}(\mathcal{B}) : (\langle f^{-1}\rangle^* C \in \mathcal{A} \Rightarrow C \in \mathcal{B}).$$

[1] We will assume that $f * \mathcal{A}$ is just a set, while it is not yet proved that it is a filter.