PROPOSITION 942. If $\exists X \in \mathcal{A} : X \subseteq A$ then:

1°. $\mathcal{A} \div A$ is a filter on A;
2°. $\mathcal{A} \div A \sim \mathcal{A}$.

PROOF.

1°. We need to prove that $\left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\}$ is a filter on A. That it is an upper set is obvious. It is non-empty because $\exists Y \in \mathcal{A} : Y \subseteq A$ and thus $A \in \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\}$. Let $P, Q \in \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\}$. Then $P, Q \subseteq A$ and $\exists P' \in \mathcal{A} : P' \subseteq P$ and $\exists Q' \in \mathcal{A} : Q' \subseteq Q$. So $P \cap Q \subseteq A$ and $P' \cap Q' \subseteq P \cap Q$. Thus $P \cap Q \in \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\}$.

2°.

$$\begin{aligned}
(\mathcal{A} \div A) \cap \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})) &= \\
\left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\} \cap \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})) &= \\
\left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\} \cap \mathscr{P} \operatorname{Base}(\mathcal{A}) &= \\
\left\{ \frac{X \in \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A}))}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\} &= \\
\mathcal{A} \cap \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})).
\end{aligned}$$

Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \cap \operatorname{Base}(\mathcal{A}) \supseteq X \in \mathcal{A}$ for some $X \in \mathcal{A}$ and

$$A \cap \operatorname{Base}(\mathcal{A}) \supseteq X \cap \operatorname{Base}(\mathcal{A}) \in \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\} = \mathcal{A} \div A.$$

□

PROPOSITION 943. $A \in \mathcal{A} \Rightarrow \mathcal{A} \div A = \mathscr{P} A \cap \mathcal{A}$.

PROOF. Let $A \in \mathcal{A}$. Then $\mathcal{A} \div A = \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\} = \left\{ \frac{X \in \mathscr{P} A}{X \in \mathcal{A}} \right\} = \mathscr{P} A \cap \mathcal{A}$. □

LEMMA 944. If $\mathcal{A} \sim \mathcal{B}$ then $\exists Y \in \mathcal{A} : Y \subseteq X \Leftrightarrow \exists Y \in \mathcal{B} : Y \subseteq X$ for every filters $\mathcal{A}$, $\mathcal{B}$, and a set X.

PROOF. We will prove $\exists Y \in \mathcal{A} : Y \subseteq X \Rightarrow \exists Y \in \mathcal{B} : Y \subseteq X$ (the other direction is similar).

We have $\mathscr{P} K \cap \mathcal{A} = \mathscr{P} K \cap \mathcal{B}$ for some set K such that $K \in \mathcal{A}$, $K \in \mathcal{B}$.

$$\exists Y \in \mathcal{A} : Y \subseteq X \Rightarrow \exists Y \in \mathscr{P} K \cap \mathcal{A} : Y \subseteq X \Rightarrow \exists Y \in \mathscr{P} K \cap \mathcal{B} : Y \subseteq X \Rightarrow \exists Y \in \mathcal{B} : Y \subseteq X.$$

□

PROPOSITION 945. If $\mathcal{A} \sim \mathcal{B}$ then $\mathcal{B} = \mathcal{A} \div \operatorname{Base}(\mathcal{B})$ for every filters $\mathcal{A}$, $\mathcal{B}$.

PROOF. $\mathscr{P} Y \cap \mathcal{A} = \mathscr{P} Y \cap \mathcal{B}$ for some set $Y \in \mathcal{A}$, $Y \in \mathcal{B}$. There exists a set $X \in \mathcal{A}$ such that $X \in \mathcal{B}$. Thus $\exists X \in \mathcal{A} : X \subseteq \operatorname{Base}(\mathcal{B})$ and so $\mathcal{A} \div \operatorname{Base}(\mathcal{B})$ is a filter.

$$\begin{aligned}
X \in \mathcal{A} \div \operatorname{Base}(\mathcal{B}) \Leftrightarrow X \in \mathscr{P} \operatorname{Base}(\mathcal{B}) \wedge \exists Y \in \mathcal{A} : Y \subseteq X \Leftrightarrow \\
X \in \mathscr{P} \operatorname{Base}(\mathcal{B}) \wedge \exists Y \in \mathcal{B} : Y \subseteq X \Leftrightarrow X \in \mathcal{B}
\end{aligned}$$

(the lemma used). □