CHAPTER 13

Orderings of filters in terms of reloids

Whilst the other chapters of this book use filters to research funcoids and reloids, here the opposite thing is discussed, the theory of reloids is used to describe properties of filters.

In this chapter the word *filter* is used to denote a filter on a set (not on an arbitrary poset) only.

13.1. Equivalent filters

DEFINITION 938. Two filters $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are equivalent ($\mathcal{A} \sim \mathcal{B}$) iff there exists a set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ *and* $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$.

PROPOSITION 939. If two filters with the same base are equivalent they are equal.

PROOF. Let $\mathcal{A}$ and $\mathcal{B}$ be two filters and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ for some set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$, and $\operatorname{Base}(\mathcal{A}) = \operatorname{Base}(\mathcal{B})$. Then

$$\mathcal{A} = (\mathscr{P}X \cap \mathcal{A}) \cup \left\{ \frac{Y \in \mathscr{P}\operatorname{Base}(\mathcal{A})}{Y \supseteq X} \right\} = (\mathscr{P}X \cap \mathcal{B}) \cup \left\{ \frac{Y \in \mathscr{P}\operatorname{Base}(\mathcal{B})}{Y \supseteq X} \right\} = \mathcal{B}.$$

□

PROPOSITION 940. $\sim$ restricted to small filters is an equivalence relation.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some small filters $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then there exist a set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ and a set Y such that $Y \in \mathcal{B}$ and $Y \in \mathcal{C}$ and $\mathscr{P}Y \cap \mathcal{B} = \mathscr{P}Y \cap \mathcal{C}$. So $X \cap Y \in \mathcal{A}$ because

$$\mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}(X \cap Y) \cap \mathcal{B} \supseteq \{X \cap Y\} \cap \mathcal{B} \ni X \cap Y.$$

Similarly we have $X \cap Y \in \mathcal{C}$. Finally

$$\mathscr{P}(X \cap Y) \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{C} = \mathscr{P}(X \cap Y) \cap \mathcal{C}.$$

□

DEFINITION 941. The *rebase* $\mathcal{A} \div A$ for a filter $\mathcal{A}$ and a set A (base) such that $\exists X \in \mathcal{A} : X \subseteq A$ is defined by the formula

$$\mathcal{A} \div A = \left\{ \frac{X \in \mathscr{P}A}{\exists Y \in \mathcal{A} : Y \subseteq X} \right\}.$$