CHAPTER 12

Total boundness of reloids

12.1. Thick binary relations

Definition 912. I will call *α-thick* and denote $\operatorname{thick}_\alpha(E)$ a **Rel**-morphism E when there exists a finite cover S of $\operatorname{Ob} E$ such that $\forall A \in S : A \times A \subseteq \operatorname{GR} E$.

Definition 913. $\operatorname{CS}(S) = \bigcup \left\{ \frac{A \times A}{A \in S} \right\}$ for a collection S of sets.

Remark 914. CS means "Cartesian squares".

Obvious 915. A **Rel**-endomorphism is α-thick iff there exists a finite cover S of $\operatorname{Ob} E$ such that $\operatorname{CS}(S) \subseteq \operatorname{GR} E$.

Definition 916. I will call *β-thick* and denote $\operatorname{thick}_\beta(E)$ a **Rel**-endomorphism E when there exists a finite set B such that $\langle E \rangle^* B = \operatorname{Ob} E$.

Proposition 917. $\operatorname{thick}_\alpha(E) \Rightarrow \operatorname{thick}_\beta(E)$.

Proof. Let $\operatorname{thick}_\alpha(E)$. Then there exists a finite cover S of the set $\operatorname{Ob} E$ such that $\forall A \in S : A \times A \subseteq \operatorname{GR} E$. Without loss of generality assume $A \neq \emptyset$ for every $A \in S$. So $A \subseteq \langle E \rangle^* \{x_A\}$ for some x_A for every $A \in S$. So

$$\langle E \rangle^* \left\{ \frac{x_A}{A \in S} \right\} = \bigcup \left\{ \frac{\langle E \rangle^* \{x_A\}}{A \in S} \right\} = \operatorname{Ob} E$$

and thus E is β-thick. $\square$

Obvious 918. Let X be a set, A and B are **Rel**-endomorphisms on X and $B \sqsupseteq A$. Then:

- $\operatorname{thick}_\alpha(A) \Rightarrow \operatorname{thick}_\alpha(B)$;
- $\operatorname{thick}_\beta(A) \Rightarrow \operatorname{thick}_\beta(B)$.

Example 919. There is a β-thick Rel-morphism which is not α-thick.

Proof. Consider the **Rel**-morphism on $[0; 1]$ with the graph on figure 1:

$$\Gamma = \left\{ \frac{(x; x)}{x \in [0; 1]} \right\} \cup \left\{ \frac{(x; 0)}{x \in [0; 1]} \right\} \cup \left\{ \frac{(0; x)}{x \in [0; 1]} \right\}.$$

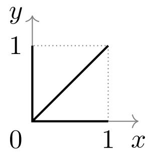

Figure 1. Thickness counterexample graph

Γ is β-thick because $\langle \Gamma \rangle^* \{0\} = [0; 1]$.

To prove that Γ is not α-thick it's enough to prove that every set A such that $A \times A \subseteq \Gamma$ is finite.