PROOF.

1°⇒2°. Let for every $a, b \in A$ there is a path between a and b in A through μ. Then $a\ (S(\mu \cap A \times A))\ b$ for every $a, b \in A$. It is possible only when $S(\mu \cap (A \times A)) \supseteq A \times A$.

3°⇒1°. For every two vertices a and b we have $a\ (S(\mu \cap A \times A))\ b$. So (by the previous theorem) for every two vertices a and b there exists a path from a to b.

3°⇒4°. Suppose $\neg(X\ [\mu \cap (A \times A)]^*\ Y)$ for some $X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg(X\ [(\mu \cap (A \times A))^n]^*\ Y)$ for every $m \in \mathbb{N}$. Consequently $\neg(X\ [S(\mu \cap (A \times A))]^*\ Y)$. So $S(\mu \cap (A \times A)) \neq A \times A$.

4°⇒3°. If $\langle S(\mu \cap (A \times A))\rangle^* \{v\} = A$ for every vertex v then $S(\mu \cap (A \times A)) = A \times A$. Consider the remaining case when $V \overset{\text{def}}{=} \langle S(\mu \cap (A \times A))\rangle^* \{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. If $\operatorname{card} A = 1$ then $S(\mu \cap (A \times A)) \supseteq \mathrm{id}_A = A \times A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V\ [\mu]^*\ W$ what is equivalent to $V\ [\mu \cap (A \times A)]^*\ W$ that is $\langle \mu \cap (A \times A)\rangle^* V \cap W \neq \emptyset$. This is impossible because

$$\langle \mu \cap (A \times A)\rangle^* V = \langle \mu \cap (A \times A)\rangle^* \langle S(\mu \cap (A \times A))\rangle^* V = \langle S_1(\mu \cap (A \times A))\rangle^* V \subseteq \langle S(\mu \cap (A \times A))\rangle^* V = V.$$

2°⇒3°. Because $S(\mu \cap (A \times A)) \subseteq A \times A$.

□

COROLLARY 886. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap (A \times A)$.

DEFINITION 887. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A.

THEOREM 888. The set A is partitioned into connected components (regarding every binary relation F).

PROOF. Consider the binary relation $a \sim b \Leftrightarrow a\ (S(F))\ b \wedge b\ (S(F))\ a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. □

PROPOSITION 889. A set is connected (regarding a binary relation) iff it has one connected component.

PROOF. Direct implication is obvious. Reverse is proved by contradiction. □

11.4. Connectedness regarding funcoids and reloids

DEFINITION 890. $S_1^*(\mu) = \bigsqcap \left\{ \frac{\uparrow^{\mathsf{RLD}} S_1(M)}{M \in \operatorname{xyGR} \mu} \right\}$ for an endoreloid μ.

DEFINITION 891. *Connectivity reloid* $S^*(\mu)$ for an endoreloid μ is defined as follows:

$$S^*(\mu) = \bigsqcap \left\{ \frac{\uparrow^{\mathsf{RLD}} S(M)}{M \in \operatorname{xyGR} \mu} \right\}.$$

Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

[1] In some math literature these two words are used interchangeably.