So a function f is uniformly continuous iff $\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\nu)} f)^{-1} \sqsubseteq \nu$.

10.2. Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν be endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu;\nu) \Leftrightarrow f \in \operatorname{Mor}(\operatorname{Ob}\mu;\operatorname{Ob}\nu) \wedge f \circ \mu \sqsubseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f \in \mathrm{C}'(\mu;\nu) \Leftrightarrow f \in \operatorname{Mor}(\operatorname{Ob}\mu;\operatorname{Ob}\nu) \wedge \mu \sqsubseteq f^{\dagger} \circ \nu \circ f;$$

$$f \in \mathrm{C}(\mu;\nu) \Leftrightarrow f \in \operatorname{Mor}(\operatorname{Ob}\mu;\operatorname{Ob}\nu) \wedge f \circ \mu \circ f^{\dagger} \sqsubseteq \nu.$$

Remark 866. In the examples (above) about funcoids and reloids the "dagger functor" is the reverse of a funcoid or reloid, that is $f^{\dagger} = f^{-1}$.

Proposition 867. Every of these three definitions of continuity forms a wide sub-precategory (wide subcategory if the original precategory is a category).

Proof.

C. Let $f \in \mathrm{C}(\mu;\nu)$, $g \in \mathrm{C}(\nu;\pi)$. Then $f \circ \mu \sqsubseteq \nu \circ f$, $g \circ \nu \sqsubseteq \pi \circ g$, $g \circ f \circ \mu \sqsubseteq g \circ \nu \circ f \sqsubseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu;\pi)$. $1_{\operatorname{Ob}\mu} \in \mathrm{C}(\mu;\mu)$ is obvious.

C′. Let $f \in \mathrm{C}'(\mu;\nu)$, $g \in \mathrm{C}'(\nu;\pi)$. Then $\mu \sqsubseteq f^{\dagger} \circ \nu \circ f$, $\nu \sqsubseteq g^{\dagger} \circ \pi \circ g$;

$$\mu \sqsubseteq f^{\dagger} \circ g^{\dagger} \circ \pi \circ g \circ f; \quad \mu \sqsubseteq (g \circ f)^{\dagger} \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu;\pi)$. $1_{\operatorname{Ob}\mu} \in \mathrm{C}'(\mu;\mu)$ is obvious.

C″. Let $f \in \mathrm{C}''(\mu;\nu)$, $g \in \mathrm{C}''(\nu;\pi)$. Then $f \circ \mu \circ f^{\dagger} \sqsubseteq \nu$, $g \circ \nu \circ g^{\dagger} \sqsubseteq \pi$;

$$g \circ f \circ \mu \circ f^{\dagger} \circ g^{\dagger} \sqsubseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^{\dagger} \sqsubseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu;\pi)$. $1_{\operatorname{Ob}\mu} \in \mathrm{C}''(\mu;\mu)$ is obvious.

□

Proposition 868. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}'(\mu;\nu) \Rightarrow f \in \mathrm{C}(\mu;\nu) \Rightarrow f \in \mathrm{C}''(\mu;\nu).$$

Proof. Let $f \in \mathrm{C}'(\mu;\nu)$. Then $\mu \sqsubseteq f^{\dagger} \circ \nu \circ f$;

$$f \circ \mu \sqsubseteq f \circ f^{\dagger} \circ \nu \circ f \sqsubseteq 1_{\operatorname{Dst} f} \circ \nu \circ f = \nu \circ f; \quad f \in \mathrm{C}(\mu;\nu).$$

Let $f \in \mathrm{C}(\mu;\nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$;

$$f \circ \mu \circ f^{\dagger} \sqsubseteq \nu \circ f \circ f^{\dagger} \sqsubseteq \nu \circ 1_{\operatorname{Dst} f} = \nu \quad f \in \mathrm{C}''(\mu;\nu).$$

□

Proposition 869. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}''(\mu;\nu) \Rightarrow f \in \mathrm{C}(\mu;\nu) \Rightarrow f \in \mathrm{C}'(\mu;\nu).$$