CHAPTER 10

Continuous morphisms

This chapter uses the apparatus from the section "Partially ordered dagger categories".

10.1. Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \sqsubseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

10.1.1. Pretopology. Let $(A; \mathrm{cl}_A)$ and $(B; \mathrm{cl}_B)$ be preclosure spaces. Then by definition a function $f : A \to B$ is continuous iff $f \, \mathrm{cl}_A(X) \subseteq \mathrm{cl}_B(fX)$ for every $X \in \mathscr{P} A$. Let now μ and ν be endofuncoids corresponding correspondingly to cl_A and cl_B. Then the condition for continuity can be rewritten as

$$\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f.$$

10.1.2. Proximity spaces. Let μ and ν be proximity spaces (which I consider a special case of endofuncoids). By definition a function f is a proximity-continuous map (also called equicontinuous) from μ to ν iff

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu) : (X \, [\mu]^* \, Y \Rightarrow \langle f \rangle^* X \, [\nu]^* \, \langle f \rangle^* Y).$$

Equivalently transforming this formulaFiXme: shortened comparing to TEXmacs version. we get

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu) : (X \, [\mu]^* \, Y \Rightarrow \langle f \rangle \uparrow^{\mathrm{Ob}\,\nu} X \, [\nu] \, \langle f \rangle \uparrow^{\mathrm{Ob}\,\nu} Y);$$

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu) : (X \, [\mu]^* \, Y \Rightarrow \uparrow^{\mathrm{Ob}\,\nu} X \left[(\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} \right] f \uparrow^{\mathrm{Ob}\,\nu} Y);$$

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu) : (X \, [\mu]^* \, Y \Rightarrow X \left[(\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f \right]^* Y);$$

$$\mu \sqsubseteq (\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f.$$

So a function f is proximity continuous iff $\mu \sqsubseteq (\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f)^{-1} \circ \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f$.

10.1.3. Uniform spaces. Uniform spaces are a special case of endoreloids.

Let μ and ν be uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \varepsilon \in \mathrm{GR}\,\nu \exists \delta \in \mathrm{GR}\,\nu \forall (x; y) \in \delta : (fx; fy) \in \varepsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \mathrm{GR}\,\nu \exists \delta \in \mathrm{GR}\,\mu \forall (x; y) \in \delta : \{(fx; fy)\} \subseteq \epsilon;$$

$$\forall \epsilon \in \mathrm{GR}\,\nu \exists \delta \in \mathrm{GR}\,\mu \forall (x; y) \in \delta : f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon;$$

$$\forall \epsilon \in \mathrm{GR}\,\nu \exists \delta \in \mathrm{GR}\,\mu : f \circ \delta \circ f^{-1} \subseteq \epsilon;$$

$$\forall \epsilon \in \mathrm{GR}\,\nu : \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f)^{-1} \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} \epsilon;$$

$$\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)} f)^{-1} \sqsubseteq \nu.$$