$$\bigsqcup\left\{\frac{g\circ F}{F\in\operatorname{atoms} f}\right\}=g\circ f\Leftrightarrow$$
$$\forall x\in\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g):\left(x\not\asymp g\circ f\Leftrightarrow x\not\asymp\bigsqcup\left\{\frac{g\circ F}{F\in\operatorname{atoms} f}\right\}\right)\Leftarrow$$
$$\forall x\in\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g):(x\not\asymp g\circ f\Leftrightarrow\exists F\in\operatorname{atoms} f:x\not\asymp g\circ F)\Leftrightarrow$$
$$\forall x\in\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g):(g^{-1}\circ x\not\asymp f\Leftrightarrow\exists F\in\operatorname{atoms} f:g^{-1}\circ x\not\asymp F)$$

what is obviously true.

Corollary 748. If f and g are composable reloids, then

$$g\circ f=\bigsqcup\left\{\frac{G\circ F}{F\in\operatorname{atoms} f,G\in\operatorname{atoms} g}\right\}.$$

Proof. $g\circ f=\bigsqcup\left\{\frac{g\circ F}{F\in\operatorname{atoms} f}\right\}=\bigsqcup\left\{\frac{\bigsqcup\left\{\frac{G\circ F}{G\in\operatorname{atoms} g}\right\}}{F\in\operatorname{atoms} f}\right\}=\bigsqcup\left\{\frac{G\circ F}{F\in\operatorname{atoms} f,G\in\operatorname{atoms} g}\right\}$. □

7.3. Reloidal product of filters

Definition 749. Reloidal product of filters $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\stackrel{\text{def}}{=}\bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))}(A\times B)}{A\in\mathcal{A},B\in\mathcal{B}}\right\}.$$

Obvious 750. $\uparrow^U A\times^{\mathsf{RLD}}\uparrow^V B=\uparrow^{\mathsf{RLD}(U;V)}(A\times B)$ for every sets $A\subseteq U$, $B\subseteq V$.

Theorem 751. $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}=\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}$ for every filters $\mathcal{A}$ and $\mathcal{B}$.

Proof. Obviously $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsupseteq\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}$.

Reversely, let $K\in\operatorname{GR}\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}$. Then $K\in\operatorname{GR}(a\times^{\mathsf{RLD}}b)$ for every $a\in\operatorname{atoms}\mathcal{A}$, $b\in\operatorname{atoms}\mathcal{B}$. $K\supseteq X_a\times Y_b$ for some $X_a\in a$, $Y_b\in b$;

$$K\supseteq\bigcup\left\{\frac{X_a\times Y_b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}=\bigcup\left\{\frac{X_a}{a\in\operatorname{atoms}\mathcal{A}}\right\}\times\bigcup\left\{\frac{Y_b}{b\in\operatorname{atoms}\mathcal{B}}\right\}\supseteq A\times B$$

where $A\in\mathcal{A}$, $B\in\mathcal{B}$; $K\in\operatorname{GR}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$. □

Theorem 752. If $\mathcal{A}_0,\mathcal{A}_1\in\mathfrak{F}(A)$, $\mathcal{B}_0,\mathcal{B}_1\in\mathfrak{F}(B)$ for some sets A, B then

$$(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0)\sqcap(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)=(\mathcal{A}_0\sqcap\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\sqcap\mathcal{B}_1).$$

Proof.

$$(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0)\sqcap(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)=$$
$$\left\{\frac{\uparrow^{\mathsf{RLD}}(P\sqcap Q)}{P\in\operatorname{xyGR}(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0),Q\in\operatorname{xyGR}(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)}\right\}=$$
$$\left\{\frac{\uparrow^{\mathsf{RLD}(A;B)}((A_0\times B_0)\cap(A_1\times B_1))}{A_0\in\mathcal{A}_0,B_0\in\mathcal{B}_0,A_1\in\mathcal{A}_1,B_1\in\mathcal{B}_1}\right\}=$$
$$\left\{\frac{\uparrow^{\mathsf{RLD}(A;B)}((A_0\cap A_1)\times(B_0\times B_1))}{A_0\in\mathcal{A}_0,B_0\in\mathcal{B}_0,A_1\in\mathcal{A}_1,B_1\in\mathcal{B}_1}\right\}=$$
$$\left\{\frac{\uparrow^{\mathsf{RLD}(A;B)}(K\times L)}{K\in\mathcal{A}_0\sqcap\mathcal{A}_1,L\in\mathcal{B}_0\sqcap\mathcal{B}_1}\right\}=$$
$$(\mathcal{A}_0\sqcap\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\sqcap\mathcal{B}_1).$$