THEOREM 681.

1°. A funcoid f is complete iff there exists a function $G : \operatorname{Src} f \to \mathfrak{F}(\operatorname{Dst} f)$ such that

$$f = \bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} G(\alpha)}{\alpha \in \operatorname{Src} f} \right\}. \tag{12}$$

2°. A funcoid f is co-complete iff there exists a function $G : \operatorname{Dst} f \to \mathfrak{F}(\operatorname{Src} f)$ such that

$$f = \bigsqcup \left\{ \frac{G(\alpha) \times^{\mathsf{FCD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}}{\alpha \in \operatorname{Dst} f} \right\}.$$

PROOF. We will prove only the first as the second is symmetric.

⇒. Let f be complete. Then take

$$G(\alpha) = \bigsqcup \left\{ \frac{b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)}}{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} b \sqsubseteq f} \right\}$$

and we have (12) obviously.

⇐. Let (12) hold. Then $G(\alpha) = \bigsqcup \operatorname{atoms} G(\alpha)$ and thus

$$f = \bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} b}{\alpha \in \operatorname{Src} f, b \in \operatorname{atoms} G(\alpha)} \right\}$$

and so f is complete.

□

THEOREM 682.

1°. For a complete funcoid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$ such that

$$f = \bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} F(\alpha)}{\alpha \in \operatorname{Src} f} \right\}.$$

2°. For a co-complete funcoid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Src} f)^{\operatorname{Dst} f}$ such that

$$f = \bigsqcup \left\{ \frac{F(\alpha) \times^{\mathsf{FCD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}}{\alpha \in \operatorname{Dst} f} \right\}.$$

PROOF. We will prove only the first as the second is similar. Let

$$f = \bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} F(\alpha)}{\alpha \in \operatorname{Src} f} \right\} = \bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} G(\alpha)}{\alpha \in \operatorname{Src} f} \right\}$$

for some $F, G \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$\langle f \rangle^* \{\beta\} = \bigsqcup \left\{ \frac{\langle \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} F(\alpha) \rangle^* \{\beta\}}{\alpha \in \operatorname{Src} f} \right\} = F(\beta).$$

Similarly $\langle f \rangle^* \{\beta\} = G(\beta)$. So $F(\beta) = G(\beta)$. □

6.12. Funcoids corresponding to pretopologies

Let Δ be a pretopology on a set U and cl the preclosure corresponding to it (see theorem 543).

Both induce a funcoid, I will show that these two funcoids are reverse of each other:

THEOREM 683. Let f be a complete funcoid defined by the formula $\langle f \rangle^* \{x\} = \Delta(x)$ for every $x \in U$, let g be a co-complete funcoid defined by the formula $\langle g \rangle^* X = \uparrow^U \operatorname{cl}(X)$ for every $X \in \mathscr{P} U$. Then $g = f^{-1}$.