Thus $\bigsqcup\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle^*S = \mathcal{A}\times^{\mathsf{FCD}}\bigsqcup S$ and $\bigsqcap\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle^*S = \mathcal{A}\times^{\mathsf{FCD}}\bigsqcap S$.
If $\mathcal{A}\neq\bot^{\mathfrak{F}(A)}$ then obviously the function $\mathcal{A}\times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a funcoidal product of filters) funcoid (of atomic width).

PROPOSITION 649. If f is a funcoid and a is an atomic filter on $\operatorname{Src} f$ then

$$f|_a = a\times^{\mathsf{FCD}}\langle f\rangle a.$$

PROOF. Let $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$.

$$\mathcal{X}\not\asymp a\Rightarrow\langle f|_a\rangle\mathcal{X} = \langle f\rangle a,\quad \mathcal{X}\asymp a\Rightarrow\langle f|_a\rangle\mathcal{X} = \bot^{\mathfrak{F}(\operatorname{Dst} f)}.$$

□

6.10. Atomic funcoids

THEOREM 650. An $f\in\mathsf{FCD}(A;B)$ is an atom of the lattice $\mathsf{FCD}(A;B)$ (for some sets A, B) iff it is a funcoidal product of two ultrafilters.

PROOF.

$\Rightarrow$. Let $f\in\mathsf{FCD}(A;B)$ be an atom of the lattice $\mathsf{FCD}(A;B)$. Let's get elements $a\in\operatorname{atoms}\operatorname{dom} f$ and $b\in\operatorname{atoms}\langle f\rangle a$. Then for every $\mathcal{X}\in\mathfrak{F}(A)$

$$\mathcal{X}\asymp a\Rightarrow\left\langle a\times^{\mathsf{FCD}}b\right\rangle\mathcal{X} = \bot^{\mathfrak{F}(B)}\sqsubseteq\langle f\rangle\mathcal{X},\quad \mathcal{X}\not\asymp a\Rightarrow\left\langle a\times^{\mathsf{FCD}}b\right\rangle\mathcal{X} = b\sqsubseteq\langle f\rangle\mathcal{X}.$$

So $a\times^{\mathsf{FCD}}b\sqsubseteq f$; because f is atomic we have $f = a\times^{\mathsf{FCD}}b$.

$\Leftarrow$. Let $a\in\operatorname{atoms}^{\mathfrak{F}(A)}$, $b\in\operatorname{atoms}^{\mathfrak{F}(B)}$, $f\in\mathsf{FCD}(A;B)$. If $b\asymp\langle f\rangle a$ then $\neg(a\ [f]\ b)$, $f\asymp a\times^{\mathsf{FCD}}b$; if $b\sqsubseteq\langle f\rangle a$ then $\forall\mathcal{X}\in\mathfrak{F}(A):(\mathcal{X}\not\asymp a\Rightarrow\langle f\rangle\mathcal{X}\sqsupseteq b)$, $f\sqsupseteq a\times^{\mathsf{FCD}}b$. Consequently $f\asymp a\times^{\mathsf{FCD}}b\vee f\sqsupseteq a\times^{\mathsf{FCD}}b$; that is $a\times^{\mathsf{FCD}}b$ is an atom.

□

THEOREM 651. The lattice $\mathsf{FCD}(A;B)$ is atomic (for every sets A, B).

PROOF. Let f be a non-empty funcoid from A to B. Then $\operatorname{dom} f\neq\bot^{\mathfrak{F}(A)}$, thus by the theorem 474 there exists $a\in\operatorname{atoms}\operatorname{dom} f$. So $\langle f\rangle a\neq\bot^{\mathfrak{F}(B)}$ thus it exists $b\in\operatorname{atoms}\langle f\rangle a$. Finally the atomic funcoid $a\times^{\mathsf{FCD}}b\sqsubseteq f$. □

THEOREM 652. The lattice $\mathsf{FCD}(A;B)$ is separable (for every sets A, B).

PROOF. Let $f,g\in\mathsf{FCD}(A;B)$, $f\sqsubset g$. Then there exists $a\in\operatorname{atoms}^{\mathfrak{F}(A)}$ such that $\langle f\rangle a\sqsubset\langle g\rangle a$. So because the lattice $\mathfrak{F}(B)$ is atomically separable, there exists $b\in\operatorname{atoms}^{\mathfrak{F}(B)}$ such that $\langle f\rangle a\sqcap b = \bot^{\mathfrak{F}(B)}$ and $b\sqsubseteq\langle g\rangle a$. For every $x\in\operatorname{atoms}^{\mathfrak{F}(A)}$

$$\langle f\rangle a\sqcap\left\langle a\times^{\mathsf{FCD}}b\right\rangle a = \langle f\rangle a\sqcap b = \bot^{\mathfrak{F}(B)},$$

$$x\neq a\Rightarrow\langle f\rangle x\sqcap\left\langle a\times^{\mathsf{FCD}}b\right\rangle x = \langle f\rangle x\sqcap\bot^{\mathfrak{F}(B)} = \bot^{\mathfrak{F}(B)}.$$

Thus $\langle f\rangle x\sqcap\left\langle a\times^{\mathsf{FCD}}b\right\rangle x = \bot^{\mathfrak{F}(B)}$ and consequently $f\asymp a\times^{\mathsf{FCD}}b$.

$$\left\langle a\times^{\mathsf{FCD}}b\right\rangle a = b\sqsubseteq\langle g\rangle a,$$

$$x\neq a\Rightarrow\left\langle a\times^{\mathsf{FCD}}b\right\rangle x = \bot^{\mathfrak{F}(B)}\sqsubseteq\langle g\rangle x.$$

Thus $\left\langle a\times^{\mathsf{FCD}}b\right\rangle x\sqsubseteq\langle g\rangle x$ and consequently $a\times^{\mathsf{FCD}}b\sqsubseteq g$.
So the lattice $\mathsf{FCD}(A;B)$ is separable by the theorem 173. □

COROLLARY 653. The lattice $\mathsf{FCD}(A;B)$ is:

1°. separable;
2°. atomically separable;