THEOREM 646. Let A, B be sets. If $S \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ then

$$\bigsqcap \left\{ \frac{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}}{(\mathcal{A}; \mathcal{B}) \in S} \right\} = \bigsqcap \operatorname{dom} S \times^{\mathsf{FCD}} \bigsqcap \operatorname{im} S.$$

PROOF. If $x \in \operatorname{atoms}^{\mathfrak{F}(A)}$ then by theorem 635

$$\left\langle \bigsqcap \left\{ \frac{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}}{(\mathcal{A}; \mathcal{B}) \in S} \right\} \right\rangle x = \bigsqcap \left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}; \mathcal{B}) \in S} \right\}.$$

If $x \not\asymp \bigsqcap \operatorname{dom} S$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S : (x \sqcap \mathcal{A} \neq \bot^{\mathfrak{F}(A)} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$

$$\left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}; \mathcal{B}) \in S} \right\} = \operatorname{im} S;$$

if $x \asymp \bigsqcap \operatorname{dom} S$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S : (x \sqcap \mathcal{A} = \bot^{\mathfrak{F}(A)} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \bot^{\mathfrak{F}(B)});$$

$$\left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}; \mathcal{B}) \in S} \right\} \ni \bot^{\mathfrak{F}(B)}.$$

So

$$\left\langle \bigsqcap \left\{ \frac{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}}{(\mathcal{A}; \mathcal{B}) \in S} \right\} \right\rangle x = \begin{cases} \bigsqcap \operatorname{im} S & \text{if } x \not\asymp \bigsqcap \operatorname{dom} S \\ \bot^{\mathfrak{F}(B)} & \text{if } x \asymp \bigsqcap \operatorname{dom} S. \end{cases}$$

From this the statement of the theorem follows. □

COROLLARY 647. For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}(B)$ (for every sets A, B)

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

PROOF. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigsqcap \{ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \}$ what is by the last theorem equal to $(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1)$. □

THEOREM 648. If A, B are sets and $\mathcal{A} \in \mathfrak{F}(A)$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism from the lattice $\mathfrak{F}(B)$ to the lattice $\mathsf{FCD}(A; B)$, if also $\mathcal{A} \neq \bot^{\mathfrak{F}(A)}$ then it is an order embedding.

PROOF. Let $S \in \mathscr{P}\mathfrak{F}(B)$, $X \in \mathscr{P}A$, $x \in \operatorname{atoms}^{\mathfrak{F}(A)}$.

$$\left\langle \bigsqcup \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle^* S \right\rangle^* X =$$

$$\bigsqcup \left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle^* X}{\mathcal{B} \in S} \right\} =$$

$$\begin{cases} \bigsqcup S & \text{if } X \in \partial \mathcal{A} \\ \bot^{\mathfrak{F}(B)} & \text{if } X \notin \partial \mathcal{A} \end{cases} =$$

$$\left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup S \right\rangle^* X;$$

$$\left\langle \bigsqcap \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle^* S \right\rangle x =$$

$$\bigsqcap \left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{\mathcal{B} \in S} \right\} =$$

$$\begin{cases} \bigsqcap S & \text{if } x \not\asymp \mathcal{A} \\ \bot^{\mathfrak{F}(B)} & \text{if } x \asymp \mathcal{A}. \end{cases}$$