□

THEOREM 628. $\operatorname{im} f = \bigsqcap \langle \operatorname{im} \rangle^* \operatorname{up} f$ and $\operatorname{dom} f = \bigsqcap \langle \operatorname{dom} \rangle^* \operatorname{up} f$ for every funcoid f.

PROOF.

$$\begin{aligned} \operatorname{im} f &= \\ \langle f \rangle \top^{\mathfrak{F}(\operatorname{Src} f)} &= \\ \bigsqcap \left\{ \frac{\langle F \rangle \top^{\mathfrak{F}(\operatorname{Src} f)}}{F \in \operatorname{up} f} \right\} &= \\ \bigsqcap \left\{ \frac{\operatorname{im} F}{F \in \operatorname{up} f} \right\} &= \\ \bigsqcap \langle \operatorname{im} \rangle^* \operatorname{up} f. \end{aligned}$$

The second formula follows from symmetry. □

PROPOSITION 629. For every composable funcoids f, g:
1°. If $\operatorname{im} f \sqsupseteq \operatorname{dom} g$ then $\operatorname{im}(g \circ f) = \operatorname{im} g$.
2°. If $\operatorname{im} f \sqsubseteq \operatorname{dom} g$ then $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

PROOF.
1°.

$$\begin{aligned} \operatorname{im}(g \circ f) &= \\ \langle g \circ f \rangle \top^{\mathfrak{F}(\operatorname{Src} f)} &= \\ \langle g \rangle \langle f \rangle \top^{\mathfrak{F}(\operatorname{Src} f)} &= \\ \langle g \rangle \operatorname{im} f &= \\ \langle g \rangle (\operatorname{im} f \sqcap \operatorname{dom} g) &= \\ \langle g \rangle \operatorname{dom} g &= \\ \langle g \rangle \top^{\mathfrak{F}(\operatorname{Src} g)} &= \\ \operatorname{im} g. \end{aligned}$$

2°. $\operatorname{dom}(g \circ f) = \operatorname{im}(f^{-1} \circ g^{-1})$ what by proved above is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$.

□

LEMMA 630. $\lambda \mathcal{B} \in \mathfrak{F}(B) : \top^{\mathfrak{F}} \times^{\mathsf{FCD}} \mathcal{B}$ is an upper adjoint of $\lambda f \in \mathsf{FCD}(A; B) : \operatorname{im} f$ (for every sets A, B).

PROOF. We need to prove $\operatorname{im} f \sqsubseteq \mathcal{B} \Leftrightarrow f \sqsubseteq \top^{\mathfrak{F}} \times^{\mathsf{FCD}} \mathcal{B}$ what is obvious. □

COROLLARY 631. Image and domain of funcoids preserve joins.

PROOF. By properties of Galois connections and duality. □

6.7. Categories of funcoids

I will define two categories, the *category of funcoids* and the *category of funcoid triples*.

The *category of funcoids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{FCD}(A; B)$.
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.