6.5. More on composition of funcoids

PROPOSITION 609. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for every composable funcoids f and g.

PROOF. For every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$ we have

$$\begin{aligned} &\mathcal{X}\ [g \circ f]\ \mathcal{Y} \Leftrightarrow \\ &\mathcal{Y} \sqcap \langle g \circ f \rangle \mathcal{X} \neq \bot^{\mathfrak{F}(\operatorname{Dst} g)} \Leftrightarrow \\ &\mathcal{Y} \sqcap \langle g \rangle \langle f \rangle \mathcal{X} \neq \bot^{\mathfrak{F}(\operatorname{Dst} g)} \Leftrightarrow \\ &\langle f \rangle \mathcal{X}\ [g]\ \mathcal{Y} \Leftrightarrow \\ &\mathcal{X}\ ([g] \circ \langle f \rangle)\ \mathcal{Y} \end{aligned}$$

and

$$\begin{aligned} &[g \circ f] = \\ &\left[(f^{-1} \circ g^{-1})^{-1}\right] = \\ &\left[f^{-1} \circ g^{-1}\right]^{-1} = \\ &\left(\left[f^{-1}\right] \circ \left\langle g^{-1} \right\rangle\right)^{-1} = \\ &\left\langle g^{-1} \right\rangle^{-1} \circ [f]. \end{aligned}$$

□

COROLLARY 610. $[f] = [\operatorname{id}_{\operatorname{Dst} f}] \circ \langle f \rangle$ for every funcoid f.FiXme: Identity funcoid is defined *below*.

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x\ (g \circ f)\ z \Leftrightarrow \exists y : (x\ f\ y \wedge y\ g\ z)$ for every x and z and every binary relations f and g.

THEOREM 611. For every sets A, B, C and $f \in \mathsf{FCD}(A; B)$, $g \in \mathsf{FCD}(B; C)$ and $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Z} \in \mathfrak{F}(C)$

$$\mathcal{X}\ [g \circ f]\ \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}(B)} : (\mathcal{X}\ [f]\ y \wedge y\ [g]\ \mathcal{Z}).$$

PROOF.

$$\begin{aligned} &\exists y \in \operatorname{atoms}^{\mathfrak{F}(B)} : (\mathcal{X}\ [f]\ y \wedge y\ [g]\ \mathcal{Z}) \Leftrightarrow \\ &\exists y \in \operatorname{atoms}^{\mathfrak{F}(B)} : (\mathcal{Z} \sqcap \langle g \rangle y \neq \bot^{\mathfrak{F}(C)} \wedge y \sqcap \langle f \rangle \mathcal{X} \neq \bot^{\mathfrak{F}(B)}) \Leftrightarrow \\ &\exists y \in \operatorname{atoms}^{\mathfrak{F}(B)} : (\mathcal{Z} \sqcap \langle g \rangle y \neq \bot^{\mathfrak{F}(C)} \wedge y \sqsubseteq \langle f \rangle \mathcal{X}) \Rightarrow \\ &\mathcal{Z} \sqcap \langle g \rangle \langle f \rangle \mathcal{X} \neq \bot^{\mathfrak{F}(C)} \Leftrightarrow \\ &\mathcal{X}\ [g \circ f]\ \mathcal{Z}. \end{aligned}$$

Reversely, if $\mathcal{X}\ [g \circ f]\ \mathcal{Z}$ then $\langle f \rangle \mathcal{X}\ [g]\ \mathcal{Z}$, consequently there exists $y \in \operatorname{atoms} \langle f \rangle \mathcal{X}$ such that $y\ [g]\ \mathcal{Z}$; we have $\mathcal{X}\ [f]\ y$. □

THEOREM 612. For every sets A, B, C

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \mathsf{FCD}(A; B)$, $f \in \mathsf{FCD}(B; C)$;
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $g, h \in \mathsf{FCD}(B; C)$, $f \in \mathsf{FCD}(A; B)$.

PROOF. I will prove only the first equality because the other is analogous.