Proof. $\langle f\rangle \bigsqcap S \sqsubseteq \langle f\rangle X$ for every $X \in S$ and thus $\langle f\rangle \bigsqcap S \sqsubseteq \bigsqcap\langle\langle f\rangle\rangle^* S$.
By properties of generalized filter bases:

$$\begin{aligned} \langle f\rangle \bigsqcap S &= \\ \bigsqcap\left\langle\langle f\rangle^*\right\rangle^* \bigsqcap S &= \\ \bigsqcap\left\langle\langle f\rangle^*\right\rangle^* \left\{\frac{X}{\exists \mathcal{P} \in S : X \in \mathcal{P}}\right\} &= \\ \bigsqcap\left\{\frac{\langle f\rangle^* X}{\exists \mathcal{P} \in S : X \in \mathcal{P}}\right\} &\sqsupseteq \\ \bigsqcap\left\{\frac{\langle f\rangle \mathcal{P}}{\mathcal{P} \in S}\right\} &= \\ \bigsqcap\langle\langle f\rangle\rangle^* S. \end{aligned}$$

□

6.4. Lattices of funcoids

Definition 600. $f \sqsubseteq g \overset{\text{def}}{=} [f]\subseteq[g]$ for $f, g \in \mathsf{FCD}(A; B)$ for every sets A, B.

Thus every $\mathsf{FCD}(A; B)$ is a poset. (It's taken into account that $[f]\neq[g]$ when $f \neq g$.)

We will consider filtrators (*filtrators of funcoids*) whose base is $\mathsf{FCD}(A; B)$ and whose core are principal funcoids from A to B.

Lemma 601. $\langle f\rangle^* X = \bigsqcap\left\{\frac{\langle F\rangle^* X}{F \in \operatorname{up} f}\right\}$ for every funcoid f and set $X \in \mathscr{P}(\operatorname{Src} f)$.

Proof. Obviously $\langle f\rangle^* X \sqsubseteq \bigsqcap\left\{\frac{\langle F\rangle^* X}{F \in \operatorname{up} f}\right\}$.
Let $B \in \langle f\rangle^* X$. Let $F_B = X \times B \sqcup ((\operatorname{Src} f) \setminus X) \times \operatorname{Dst} f$.
$\langle F_B\rangle^* X = B$.
Let $P \in \mathscr{P}(\operatorname{Src} f)$. We have

$$\emptyset \neq P \subseteq X \Rightarrow \uparrow^{\operatorname{Dst} f} \langle F_B\rangle^* P = \uparrow^{\operatorname{Dst} f} B \sqsupseteq \langle f\rangle^* P$$

and

$$\emptyset \neq P \nsubseteq X \Rightarrow \uparrow^{\operatorname{Dst} f} \langle F_B\rangle^* = \uparrow^{\operatorname{Dst} f} \operatorname{Dst} f \sqsupseteq \langle f\rangle^* P.$$

Thus $\uparrow^{\operatorname{Dst} f} \langle F_B\rangle^* P \sqsupseteq \langle f\rangle^* P$ for every P and so $\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F_B \sqsupseteq f$ that is $F_B \in \operatorname{up} f$.
Thus $\forall B \in \langle f\rangle^* X : B \in \bigsqcap\left\{\frac{\langle F\rangle^* X}{F \in \operatorname{up} f}\right\}$ because $B \in \left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F_B\right\rangle^* X$.
So $\bigsqcap\left\{\frac{\langle F\rangle^* X}{F \in \operatorname{up} f}\right\} \sqsubseteq \langle f\rangle^* X$. □

Theorem 602. $\langle f\rangle \mathcal{X} = \bigsqcap\left\{\frac{\langle F\rangle \mathcal{X}}{F \in \operatorname{up} f}\right\}$ for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.