If τ is the pretopology induced by topology π, in turn induced by a Kuratowski closure ρ, then $\tau = \rho$.

$$\begin{aligned}\mathrm{cl}_\tau(A) &= \\ \bigcap\left\{\frac{X \in \mathscr{P}U}{X \text{ is a closed set in } \pi, X \supseteq A}\right\} &= \\ \bigcap\left\{\frac{X \in \mathscr{P}U}{\mathrm{cl}_\rho(X) = X, X \supseteq A}\right\} &= \\ \bigcap\left\{\frac{\mathrm{cl}_\rho(X)}{X \in \mathscr{P}U, \mathrm{cl}_\rho(X) = X, X \supseteq \mathrm{cl}_\rho(A)}\right\} &= \\ \bigcap\left\{\frac{\mathrm{cl}_\rho(\mathrm{cl}_\rho(X))}{X = A}\right\} &= \\ \mathrm{cl}_\rho(\mathrm{cl}_\rho(A)) &= \\ \mathrm{cl}_\rho(A).\end{aligned}$$

□

5.3.1.3. *Topology induced by a metric.*

DEFINITION 560. Every metric space induces a topology in this way: A set X is open iff

$$\forall x \in X \exists \epsilon > 0 : B_r(x) \subseteq X.$$

EXERCISE 561. Prove it is really a topology and this topology is the same as the topology, induced by the pretopology, in turn induced by our metric space.

5.4. Proximity spaces

Let $(U; d)$ be metric space. We will define *distance* between sets $A, B \in \mathscr{P}U$ by the formula

$$d(A, B) = \inf\left\{\frac{d(a, b)}{a \in A, b \in B}\right\}.$$

(Here "inf" denotes infimum on the real line.)

DEFINITION 562. Sets $A, B \in \mathscr{P}U$ are *near* (denoted $A \,\delta\, B$) iff $d(A, B) = 0$.

δ defined in this way (for a metric space) is an example of proximity as defined below.

DEFINITION 563. A *proximity space* is a set $(U; \delta)$ conforming to the following axioms (for every $A, B, C \in \mathscr{P}U$):

1°. $A \cap B \neq \emptyset \Rightarrow A \,\delta\, B$;
2°. if $A \,\delta\, B$ then $A \neq \emptyset$ and $B \neq \emptyset$;
3°. $A \,\delta\, B \Rightarrow B \,\delta\, A$ (*symmetry*);
4°. $(A \cup B) \,\delta\, C \Leftrightarrow A \,\delta\, C \vee B \,\delta\, C$;
5°. $C \,\delta\, (A \cup B) \Leftrightarrow C \,\delta\, A \vee C \,\delta\, B$;
6°. $A \,\bar{\delta}\, B$ implies existence of $P, Q \in \mathscr{P}U$ with $A \,\bar{\delta}\, P$, $B \,\bar{\delta}\, Q$ and $P \cup Q = U$.

EXERCISE 564. Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

DEFINITION 565. *Quasi-proximity* is defined as the above but without the symmetry axiom.

DEFINITION 566. Closure is generated by a proximity by the following formula:

$$\mathrm{cl}(A) = \left\{\frac{a \in U}{\{a\} \,\delta\, A}\right\}.$$