Having a pretopological space $(U;\Delta)$ we define a topological space whose open sets are

$$\left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \Delta(x)} \right\}.$$

PROPOSITION 553. This really defines a topology.

PROOF. Let set $S \subseteq \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \Delta(x)} \right\}$. Then $\forall X \in S \forall x \in X : X \in \Delta(x)$. Thus

$$\forall x \in \bigcup S \exists X \in S : X \in \Delta(x)$$

and so $\forall x \in \bigcup S : \bigcup S \in \Delta(x)$. So $\bigcup S$ is an open set.

Let now $A_0, \ldots, A_n \in \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \Delta(x)} \right\}$ for $n \in \mathbb{N}$. Then $\forall x \in A_i : A_i \in \Delta(x)$ and so

$$\forall x \in A_0 \cap \cdots \cap A_n : A_i \in \Delta(x);$$

thus $\forall x \in A_0 \cap \cdots \cap A_n : A_0 \cap \cdots \cap A_n \in \Delta(x)$. So $A_0 \cap \cdots \cap A_n \in \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \Delta(x)} \right\}$.
That U is an open set is obvious. □

PROPOSITION 554. Topology τ defined by a pretopology and topology ρ defined by the corresponding preclosure, are the same.

PROOF. Let $A \in \mathscr{P}U$.
A is ρ-closed $\Leftrightarrow \operatorname{cl}(A) = A \Leftrightarrow \operatorname{cl}(A) \subseteq A \Leftrightarrow \forall x \in U : (A \in \partial\Delta(x) \Rightarrow x \in A)$;

$$\begin{aligned} &A \text{ is } \tau\text{-open} \Leftrightarrow \\ &\forall x \in A : A \in \Delta(x) \Leftrightarrow \\ &\forall x \in U : (x \in A \Rightarrow A \in \Delta(x)) \Leftrightarrow \\ &\forall x \in U : (x \notin U \setminus A \Rightarrow U \setminus A \notin \partial\Delta(x)). \end{aligned}$$

So ρ-closed and τ-open are negations of each other. It follows $\rho = \tau$. □

5.3.1.2. *Preclosure space induced by topological space.* We define a preclosure and a pretopology induced by a topology and then show these two are equivalent.

Having a topological space we define a preclosure space by the formula

$$\operatorname{cl}(A) = \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\}.$$

PROPOSITION 555. It is really a preclosure.

PROOF. $\operatorname{cl}(\emptyset) = \emptyset$ because $\emptyset$ is a closed set. $\operatorname{cl}(A) \supseteq A$ is obvious.

$$\begin{aligned} &\operatorname{cl}(A \cup B) = \\ &\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A \cup B} \right\} = \\ &\bigcap \left\{ \frac{X_1 \cup X_2}{X_1, X_2 \in \mathscr{P}U \text{ are closed sets}, X_1 \supseteq A, X_2 \supseteq B} \right\} = \\ &\bigcap \left\{ \frac{X_1 \in \mathscr{P}U}{X_1 \text{ is a closed set}, X_1 \supseteq A} \right\} \cup \bigcap \left\{ \frac{X_2 \in \mathscr{P}U}{X_2 \text{ is a closed set}, X_2 \supseteq B} \right\} = \\ &\operatorname{cl}(A) \cup \operatorname{cl}(B). \end{aligned}$$

Thus cl is a preclosure. □

Or: $\Delta(x) = \prod \left\{ \frac{\uparrow^U X}{X \in \mathcal{O}, x \in X} \right\}$.
It is trivially a pretopology (used the fact that $U \in \mathcal{O}$).