So there exist two different $\mathcal{F} \in K$ such that $A \in \mathcal{F}$. Consequently $\exists \mathcal{F} \in K \setminus \{\mathcal{G}\} : A \in \mathcal{F}$ that is $\bigsqcup(K \setminus \{\mathcal{G}\}) = \Omega$. □

EXAMPLE 522. There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

PROOF. Consider cofinite filter Ω on any infinite set.

Suppose K is its weak partition into ultrafilters. Then $x \asymp \bigsqcup(K \setminus \{x\})$ for some ultrafilter $x \in K$.

We have $\bigsqcup(K \setminus \{x\}) \sqsubset \bigsqcup K$ (otherwise $x \sqsubseteq \bigsqcup(K \setminus \{x\})$) what is impossible due the last lemma. □

COROLLARY 523. There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

4.6. Open problems about filters

In this section, I will formulate some conjectures about lattices of filters on a set. If a conjecture comes true, it may be generalized for more general lattices (such as, for example, lattices of filters on arbitrary lattices). I deem that the main challenge is to prove the special case about lattices of filters on a set, and generalizing the conjectures is expected to be an easy task.

4.6.1. Partitioning. Consider the complete lattice $[S]$ generated by the set S where S is a strong partition of some element a.

CONJECTURE 524. $[S] = \left\{ \frac{\bigsqcup^{\mathfrak{F}} X}{X \in \mathscr{P}S} \right\}$, where $[S]$ is the complete lattice generated by a strong partition S of filter on a set.

Consider also the similar conjecture with weak partition instead strong partition. FiXme: Formulate that conjecture and related statements explicitly.

PROPOSITION 525. Provided that the last conjecture is true, we have that $[S]$ is a complete atomic boolean lattice with the set of its atoms being S.

REMARK 526. Consequently $[S]$ is atomistic, completely distributive and isomorphic to a power set algebra (see [39]).

PROOF. Completeness of $[S]$ is obvious. Let $A \in [S]$. Then there exists $X \in \mathscr{P}S$ such that $A = \bigsqcup^{\mathfrak{F}} X$. Let $B = \bigsqcup^{\mathfrak{F}}(S \setminus X)$. Then $B \in [S]$ and $A \sqcap^{\mathfrak{F}} B = \bot^{\mathfrak{F}}$. $A \sqcup^{\mathfrak{F}} B = \bigsqcup^{\mathfrak{F}} S$ is the greatest element of $[S]$. So we have proved that $[S]$ is a boolean lattice.

Now let prove that $[S]$ is atomic with the set of atoms being S. Let $z \in S$ and $A \in [S]$. If $A \neq z$ then either $A = \bot^{\mathfrak{F}}$ or $x \in X$ where $A = \bigsqcup^{\mathfrak{F}} X$, $X \in \mathscr{P}S$ and $x \neq z$. Because S is a partition, $\bigsqcup^{\mathfrak{F}}(X \setminus \{z\}) \sqcap^{\mathfrak{F}} z = \bot^{\mathfrak{F}}$ and $\bigsqcup^{\mathfrak{F}}(X \setminus \{z\}) \neq \bot^{\mathfrak{F}}$. So $A = \bigsqcup^{\mathfrak{F}} X = \bigsqcup^{\mathfrak{F}}(X \setminus \{z\}) \sqcup^{\mathfrak{F}} z \not\sqsubseteq z$.

Finally we will prove that elements of $[S] \setminus S$ are not atoms. Let $A \in [S] \setminus S$ and $A \neq \bot$. Then $A \sqsupseteq x \sqcup^{\mathfrak{F}} y$ where $x, y \in S$ and $x \neq y$. If A is an atom then $A = x = y$ what is impossible. □

PROPOSITION 527. The conjecture about the value of $[S]$ is equivalent to closedness of $\left\{ \frac{\bigsqcup^{\mathfrak{F}} X}{X \in \mathscr{P}S} \right\}$ under arbitrary meets and joins.

PROOF. If $\left\{ \frac{\bigsqcup^{\mathfrak{F}} X}{X \in \mathscr{P}S} \right\} = [S]$ then trivially $\left\{ \frac{\bigsqcup^{\mathfrak{F}} X}{X \in \mathscr{P}S} \right\}$ is closed under arbitrary meets and joins.