But this can be easily accomplished taking F having zero or one element in each of intervals to which $r_1, \ldots, r_k, s_1, \ldots, s_k$ split the real line. □

EXAMPLE 519. There exists a weak partition of a filter on a set which is not a strong partition.

PROOF. (suggested by ANDREAS BLASS) Let $\left\{ \frac{X_r}{r\in\mathbb{R}} \right\}$ an independent family of subsets of $\mathbb{N}$. We can assume $a \neq b \Rightarrow X_a \neq X_b$ due the above lemma.

Let $\mathcal{F}_a$ be a filter generated by X_a and the complements $\mathbb{N} \setminus X_b$ for all $b \in \mathbb{R}$, $b \neq a$. Independence implies that $\mathcal{F}_a \neq \bot^{\mathfrak{F}}$ (by properties of filter bases).

Let $S = \left\{ \frac{\mathcal{F}_r}{r\in\mathbb{R}} \right\}$. We will prove that S is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\} : \mathbb{N} \setminus X_a \in \mathcal{F}_b$ and therefore $\mathbb{N} \setminus X_a \in \bigsqcup^{\mathfrak{F}} \left\{ \frac{\mathcal{F}_b}{\mathbb{R}\ni b\neq a} \right\}$. Therefore $\mathcal{F}_a \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} \left\{ \frac{\mathcal{F}_b}{\mathbb{R}\ni b\neq a} \right\} = \bot^{\mathfrak{F}}$. Thus S is a weak partition.

Suppose S is a strong partition. Then for each set $Z \in \mathscr{P}\mathbb{R}$

$$\bigsqcup^{\mathfrak{F}} \left\{ \frac{\mathcal{F}_b}{b \in Z} \right\} \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} \left\{ \frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z} \right\} = \bot^{\mathfrak{F}}$$

what is equivalent to existence of $M(Z) \in \mathscr{P}\mathbb{N}$ such that

$$M(Z) \in \bigsqcup^{\mathfrak{F}} \left\{ \frac{\mathcal{F}_b}{b \in Z} \right\} \quad \text{and} \quad \mathbb{N} \setminus M(Z) \in \bigsqcup^{\mathfrak{F}} \left\{ \frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z} \right\}$$

that is

$$\forall b \in Z : M(Z) \in \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathbb{R} \setminus Z : \mathbb{N} \setminus M(Z) \in \mathcal{F}_b.$$

Suppose $Z \neq Z' \in \mathscr{P}\mathbb{N}$. Without loss of generality we may assume that some $b \in Z$ but $b \notin Z'$. Then $M(Z) \in \mathcal{F}_b$ and $\mathbb{N} \setminus M(Z') \in \mathcal{F}_b$. If $M(Z) = M(Z')$ then $\mathcal{F}_b = \bot^{\mathfrak{F}}$ what contradicts to the above.

So M is an injective function from $\mathscr{P}\mathbb{R}$ to $\mathscr{P}\mathbb{N}$ what is impossible due cardinality issues. □

LEMMA 520. (by NIELS DIEPEVEEN, with help of KARL KRONENFELD) Let K be a collection of free ultrafilters. We have $\bigsqcup K = \Omega$ iff $\exists \mathcal{G} \in K : A \in \mathcal{G}$ for every infinite set A.

PROOF.

$\Rightarrow$. Suppose $\bigsqcup K = \Omega$ and let A be a set such that $\nexists \mathcal{G} \in K : A \in \mathcal{G}$. Let's prove A is finite.

Really, $\forall \mathcal{G} \in K : \mathfrak{U} \setminus A \in \mathcal{G}$; $\mathfrak{U} \setminus A \in \Omega$; A is finite.

$\Leftarrow$. Let $\exists \mathcal{G} \in K : A \in \mathcal{G}$. Suppose A is a set in $\bigsqcup K$.

To finish the proof it's enough to show that $\mathfrak{U} \setminus A$ is finite.

Suppose $\mathfrak{U} \setminus A$ is infinite. Then $\exists \mathcal{G} \in K : \mathfrak{U} \setminus A \in \mathcal{G}$; $\exists \mathcal{G} \in K : A \notin \mathcal{G}$; $A \notin \bigsqcup K$, contradiction.

□

LEMMA 521. (by NIELS DIEPEVEEN) If K is a non-empty set of ultrafilters such that $\bigsqcup K = \Omega$, then for every $\mathcal{G} \in K$ we have $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$.

PROOF. $\exists \mathcal{F} \in K : A \in \mathcal{F}$ for every infinite set A.

The set A can be partitioned into two infinite sets A_1, A_2.

Take $\mathcal{F}_1, \mathcal{F}_2 \in K$ such that $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$.

$\mathcal{F}_1 \neq \mathcal{F}_2$ because otherwise A_1 and A_2 are not disjoint.

Obviously $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$.