PROOF. Let M be the set of finite subsets of U.

$$\begin{aligned} \uparrow X \sqcap^{\mathfrak{F}} \Omega \sqsubseteq \uparrow Y \sqcap^{\mathfrak{F}} \Omega &\Leftrightarrow \\ \left\{ \frac{X \cap K_X}{K_X \in \Omega} \right\} \supseteq \left\{ \frac{Y \cap K_Y}{K_Y \in \Omega} \right\} &\Leftrightarrow \\ \forall K_Y \in \Omega \exists K_X \in \Omega : Y \cap K_Y = X \cap K_X &\Leftrightarrow \\ \forall L_Y \in M \exists L_X \in M : Y \setminus L_Y = X \setminus L_X &\Leftrightarrow \\ \forall L_Y \in M : X \setminus (Y \setminus L_Y) \in M &\Leftrightarrow \\ X \setminus Y \in M&. \end{aligned}$$

□

EXAMPLE 515. There exists a filter $\mathcal{A}$ on a set U such that $(\mathscr{P}U)/\sim$ and $Z(D\mathcal{A})$ are not complete lattices.

PROOF. Due to the isomorphism it is enough to prove for $(\mathscr{P}U)/\sim$.

Let take $U = \mathbb{N}$ and $\mathcal{A} = \Omega$ be the Fréchet filter on $\mathbb{N}$.

Partition $\mathbb{N}$ into infinitely many infinite sets $A_0, A_1, \ldots$. To withhold our example we will prove that the set $\{[A_0], [A_1], \ldots\}$ has no supremum in $(\mathscr{P}U)/\sim$.

Let $[X]$ be an upper bound of $[A_0], [A_1], \ldots$ that is $\forall i \in \mathbb{N} : \uparrow X \sqcap^{\mathfrak{F}} \Omega \sqsupseteq \uparrow A_i \sqcap^{\mathfrak{F}} \Omega$ that is $A_i \setminus X$ is finite. Consequently X is infinite. So $X \cap A_i \neq \emptyset$.

Choose for every $i \in \mathbb{N}$ some $z_i \in X \cap A_i$. The $\{z_0, z_1, \ldots\}$ is an infinite subset of X (take into account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \ldots\}$. Then $\uparrow Y \sqcap^{\mathfrak{F}} \Omega \sqsupseteq \uparrow A_i \sqcap^{\mathfrak{F}} \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \ldots\}$.

Suppose $\uparrow Y \sqcap^{\mathfrak{F}} \Omega = \uparrow X \sqcap^{\mathfrak{F}} \Omega$. Then $Y \setminus X$ is finite what is not true. So $\uparrow Y \sqcap^{\mathfrak{F}} \Omega \sqsubset \uparrow X \sqcap^{\mathfrak{F}} \Omega$ that is $[Y]$ is below $[X]$. □

4.5.1. Weak and Strong Partition.

DEFINITION 516. A family S of subsets of a countable set is *independent* iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.

LEMMA 517. The "infinite" at the end of the definition could be equivalently replaced with "nonempty" if we assume that S is infinite.

PROOF. Suppose that some sets from the above definition has a finite intersection J of cardinality n. Then (thanks S is infinite) get one more set $X \in S$ and we have $J \cap X \neq \emptyset$ and $J \cap (\mathbb{N} \setminus X) \neq \emptyset$. So $\operatorname{card}(J \cap X) < n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. □

LEMMA 518. There exists an independent family on $\mathbb{N}$ of cardinality $\mathfrak{c}$.

PROOF. Let C be the set of finite subsets of $\mathbb{Q}$. Since $\operatorname{card} C = \operatorname{card} \mathbb{N}$, it suffices to find $\mathfrak{c}$ independent subsets of C. For each $r \in \mathbb{R}$ let

$$E_r = \left\{ \frac{F \in C}{\operatorname{card}(F \cap (-\infty; r)) \text{ is even}} \right\}.$$

All E_{r_1} and E_{r_2} are distinct for distinct $r_1, r_2 \in \mathbb{R}$ since we may consider $F = \{r'\} \in C$ where a rational number r' is between r_1 and r_2 and thus F is a member of exactly one of the sets E_{r_1} and E_{r_2}. Thus $\operatorname{card}\left\{ \frac{E_r}{r \in \mathbb{R}} \right\} = \mathfrak{c}$.

We will show that $\left\{ \frac{E_r}{r \in \mathbb{R}} \right\}$ is independent. Let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be distinct reals. It is enough to show that these have a nonempty intersection, that is existence of some F such that F belongs to all the E_r and none of E_s.