Proof. Let
$$X' = \left\{ \frac{(P;Q)}{P \in \mathscr{P}X \text{ is finite}, Q \in \mathscr{P}\mathscr{P}P} \right\}.$$
It's easy to show that $\operatorname{card} X' = \operatorname{card} X$. So it is enough to show this for X' instead of X. Let
$$\mathcal{F} = \left\{ \frac{\left\{ \frac{(P;Q)\in X'}{Y\cap P \in Q} \right\}}{Y \in \mathscr{P}X} \right\}.$$
To finish the proof we show that for every disjoint finite $Y_+ \in \mathscr{P}\mathscr{P}X$ and finite $Y_- \in \mathscr{P}\mathscr{P}X$ there exist $(P;Q) \in X'$ such that
$$\forall Y \in Y_+ : (P;Q) \in \left\{ \frac{(P;Q) \in X'}{Y \cap P \in Q} \right\} \quad \text{and} \quad \forall Y \in Y_- : (P;Q) \notin \left\{ \frac{(P;Q) \in X'}{Y \cap P \in Q} \right\}$$
what is equivalent to existence $(P;Q) \in X'$ such that
$$\forall Y \in Y_+ : Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_- : Y \cap P \notin Q.$$
For existence of this $(P;Q)$, it is enough existence of P such that intersections $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$.

Really, for each pair of distinct $Y_0, Y_1 \in Y_+ \cup Y_-$ choose a point which lies in one of the sets Y_0, Y_1 and not in an other, and call the set of such points P. Then $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$. □

Corollary 507. For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for arbitrary disjoint subfamilies $\mathcal{A}$ and $\mathcal{B}$ the set $\mathcal{A} \cup \left\{ \frac{X \setminus A}{A \in \mathcal{B}} \right\}$ has finite intersection property.

Theorem 508. Let X be a set. The number of ultrafilters on X is $2^{2^{\operatorname{card} X}}$ if X is infinite and $\operatorname{card} X$ if X is finite.

Proof. The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let X be infinite. From the lemma, there exists a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for every $\mathcal{G} \in \mathscr{P}\mathcal{F}$ we have $\Phi(\mathcal{F};\mathcal{G}) = \bigsqcap \langle \uparrow \rangle^* \mathcal{G} \sqcap \bigsqcap \langle \uparrow \rangle^* \left\{ \frac{X \setminus A}{A \in \mathcal{F} \setminus \mathcal{G}} \right\} \neq \bot^{\mathfrak{F}(X)}$.

This filter contains all sets from $\mathcal{G}$ and does not contain any sets from $\mathcal{F} \setminus \mathcal{G}$. So for every suitable pairs $(\mathcal{F}_0;\mathcal{G}_0)$ and $(\mathcal{F}_1;\mathcal{G}_1)$ there is $A \in \Phi(\mathcal{F}_0;\mathcal{G}_0)$ such that $\overline{A} \in \Phi(\mathcal{F}_1;\mathcal{G}_1)$. Consequently all filters $\Phi(\mathcal{F};\mathcal{G})$ are disjoint. So for every pair $(\mathcal{F};\mathcal{G})$ where $\mathcal{G} \in \mathscr{P}\mathcal{F}$ there exist a distinct ultrafilter under $\Phi(\mathcal{F};\mathcal{G})$, but the number of such pairs $(\mathcal{F};\mathcal{G})$ is $2^{2^{\operatorname{card} X}}$. Obviously the number of all filters is not above $2^{2^{\operatorname{card} X}}$. □

Corollary 509. The number of filters on $\mathfrak{U}$ is $2^{2^{\operatorname{card} X}}$ if $\mathfrak{U}$ us infinite and $2^{\operatorname{card} \mathfrak{U}}$ if $\mathfrak{U}$ is finite.

Proof. The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than $2^{2^{\operatorname{card} \mathfrak{U}}}$ collections of sets on $\mathfrak{U}$. □

4.5. Some Counter-Examples

Example 510. There exist a bounded distributive lattice which is not lattice with separable center.