CONJECTURE 497. $a \setminus^* b = a \# b$ for arbitrary filters a, b on powersets is not provable in ZF (without axiom of choice).

4.4.1. Fréchet Filter. The consideration below is about filters on a set $\mathfrak{U}$, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorphic to algebras of sets on some set $\mathfrak{U}$.

DEFINITION 498. $\Omega = \left\{ \frac{\mathfrak{U} \setminus X}{X \text{ is a finite subset of } \mathfrak{U}} \right\}$ is called either *Fréchet filter* or *cofinite filter*.

It is trivial that Fréchet filter is a filter.

PROPOSITION 499. $\operatorname{Cor} \Omega = \bot^{\mathfrak{P}}$; $\bigcap \Omega = \emptyset$.

PROOF. This can be deduced from the formula $\forall \alpha \in \mathfrak{U} \exists X \in \Omega : \alpha \notin X$. □

THEOREM 500. $\max \left\{ \frac{\mathcal{X} \in \mathfrak{F}}{\operatorname{Cor} \mathcal{X} = \bot^{\mathfrak{P}}} \right\} = \max \left\{ \frac{\mathcal{X} \in \mathfrak{F}}{\bigcap \mathcal{X} = \emptyset} \right\} = \Omega$.

PROOF. Due the last proposition, it is enough to show that $\operatorname{Cor} \mathcal{X} = \bot^{\mathfrak{P}} \Rightarrow \mathcal{X} \sqsubseteq \Omega$ for every filter $\mathcal{X}$.

Let $\operatorname{Cor} \mathcal{X} = \bot^{\mathfrak{P}}$ for some filter $\mathcal{X}$. Let $X \in \Omega$. We need to prove that $X \in \mathcal{X}$.

$X = \mathfrak{U} \setminus \{\alpha_0, \ldots, \alpha_n\}$. $\mathfrak{U} \setminus \{\alpha_i\} \in \mathcal{X}$ because otherwise $\alpha_i \in \uparrow^{-1} \operatorname{Cor} \mathcal{X}$. So $X \in \mathcal{X}$. □

THEOREM 501. $\Omega = \bigsqcup^{\mathfrak{F}} \left\{ \frac{x}{x \text{ is a non-trivial ultrafilter}} \right\}$.

PROOF. It follows from the facts that $\operatorname{Cor} x = \bot^{\mathfrak{P}}$ for every non-trivial ultrafilter x, that $\mathfrak{F}$ is an atomistic lattice, and the previous theorem. □

THEOREM 502. Cor is the lower adjoint of $\Omega \sqcup^{\mathfrak{F}} -$.

PROOF. Because both Cor and $\Omega \sqcup^{\mathfrak{F}} -$ are monotone, it is enough (theorem 108) to prove (for every filters $\mathcal{X}$ and $\mathcal{Y}$)

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{F}} \operatorname{Cor} \mathcal{X} \quad \text{and} \quad \operatorname{Cor}(\Omega \sqcup^{\mathfrak{F}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$\operatorname{Cor}(\Omega \sqcup^{\mathfrak{F}} \mathcal{Y}) = \operatorname{Cor} \Omega \sqcup^{\mathfrak{P}} \operatorname{Cor} \mathcal{Y} = \bot^{\mathfrak{P}} \sqcup^{\mathfrak{P}} \operatorname{Cor} \mathcal{Y} = \operatorname{Cor} \mathcal{Y} \sqsubseteq \mathcal{Y}$. □

$\Omega \sqcup^{\mathfrak{F}} \operatorname{Cor} \mathcal{X} \sqsupseteq \operatorname{Edg} \mathcal{X} \sqcup^{\mathfrak{F}} \operatorname{Cor} \mathcal{X} = \mathcal{X}$.

COROLLARY 503. $\operatorname{Cor} \mathcal{X} = \mathcal{X} \setminus^* \Omega$ for every filter on a set.

PROOF. By theorem 125. □

COROLLARY 504. $\operatorname{Cor} \bigsqcup^{\mathfrak{F}} S = \bigsqcup^{\mathfrak{F}} \langle \operatorname{Cor} \rangle^* S$ for any set S of filters on a powerset.FiXme: ANDREAS BLASS provided a more elementary proof of this.

4.4.2. Number of Filters on a Set.

DEFINITION 505. A collection Y of sets has finite intersection property iff intersection of any finite subcollection of Y is non-empty.

The following was borrowed from [7]. Thanks to ANDREAS BLASS for email support about his proof.

LEMMA 506. (by HAUSDORFF) For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that given any disjoint finite subfamilies $\mathcal{A}$, $\mathcal{B}$, the intersection of sets in $\mathcal{A}$ and complements of sets in $\mathcal{B}$ is nonempty.