PROOF. $(\mathfrak{F};\mathfrak{P})$ is a filtered (corollary 362), distributive (corollary 381) complete lattice filtrator (corollary 374), with finitely meet-closed core (proposition 364), with separable core (theorem 379). So we can apply the theorem 339. □

4.3.21. Complementive Filters and Factoring by a Filter.

DEFINITION 426. Let $\mathfrak{A}$ be a meet-semilattice and $\mathcal{A} \in \mathfrak{A}$. The relation $\sim$ on $\mathfrak{A}$ is defined by the formula

$$\forall X, Y \in \mathfrak{A} :(X \sim Y \Leftrightarrow X \sqcap^{\mathfrak{A}} \mathcal{A} = Y \sqcap^{\mathfrak{A}} \mathcal{A}).$$

PROPOSITION 427. The relation $\sim$ is an equivalence relation.

PROOF.

Reflexivity. Obvious.
Symmetry. Obvious.
Transitivity. Obvious.

□

DEFINITION 428. When $X, Y \in \mathfrak{Z}$ and $\mathcal{A} \in \mathfrak{F}$ we define $X \sim Y \Leftrightarrow \uparrow X \sim \uparrow Y$.

THEOREM 429. Let $\mathfrak{Z}$ be a distributive latticeFiXme: Generalize for meet-semilattices?, $\mathcal{A} \in \mathfrak{F}$. Then for every $X, Y \in \mathfrak{Z}$

$$X \sim Y \Leftrightarrow \exists A \in \mathcal{A} : X \sqcap^{\mathfrak{Z}} A = Y \sqcap^{\mathfrak{Z}} A.$$

PROOF.

$$\begin{aligned}\exists A \in \mathcal{A} : X \sqcap^{\mathfrak{Z}} A = Y \sqcap^{\mathfrak{Z}} A &\Leftrightarrow \\ \exists A \in \mathcal{A} :\uparrow X \sqcap^{\mathfrak{F}} \uparrow A = \uparrow Y \sqcap^{\mathfrak{F}} \uparrow A &\Rightarrow \\ \exists A \in \mathcal{A} :\uparrow X \sqcap^{\mathfrak{F}} \uparrow A \sqcap^{\mathfrak{F}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{F}} \uparrow A \sqcap^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\ \exists A \in \mathcal{A} :\uparrow X \sqcap^{\mathfrak{F}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\ \uparrow X \sqcap^{\mathfrak{F}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\ \uparrow X \sim \uparrow Y &\Leftrightarrow \\ X \sim Y&.\end{aligned}$$

On the other hand,

$$\begin{aligned}\uparrow X \sqcap^{\mathfrak{F}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\ \left\{ \frac{X \sqcap^{\mathfrak{Z}} A_0}{A_0 \in \mathcal{A}} \right\} = \left\{ \frac{Y \sqcap^{\mathfrak{Z}} A_1}{A_1 \in \mathcal{A}} \right\} &\Rightarrow \\ \exists A_0, A_1 \in \mathcal{A} : X \sqcap^{\mathfrak{Z}} A_0 = Y \sqcap^{\mathfrak{Z}} A_1 &\Rightarrow \\ \exists A_0, A_1 \in \mathcal{A} : X \sqcap^{\mathfrak{Z}} A_0 \sqcap^{\mathfrak{Z}} A_1 = Y \sqcap^{\mathfrak{Z}} A_0 \sqcap^{\mathfrak{Z}} A_1 &\Rightarrow \\ \exists A \in \mathcal{A} : Y \sqcap^{\mathfrak{Z}} A = X \sqcap^{\mathfrak{Z}} A&.\end{aligned}$$

□

PROPOSITION 430. The relation $\sim$ is a congruence[1] for each of the following:

1°. a meet-semilattice $\mathfrak{A}$;
2°. a distributive lattice $\mathfrak{A}$.

PROOF. Let $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ and $a_0 \sim a_1$ and $b_0 \sim b_1$.

1°. $a_0 \sqcap b_0 \sim a_1 \sqcap b_1$because $(a_0 \sqcap b_0) \sqcap \mathcal{A} = a_0 \sqcap (b_0 \sqcap \mathcal{A}) = a_0 \sqcap (b_1 \sqcap \mathcal{A}) = b_1 \sqcap (a_0 \sqcap \mathcal{A}) = b_1 \sqcap (a_1 \sqcap \mathcal{A}) = (a_1 \sqcap b_1) \sqcap \mathcal{A}$.

[1]See Wikipedia for a definition of congruence.