PROOF. $(\mathfrak{F};\mathfrak{P})$ is with finitely join-closed core by the theorem 363, $\mathfrak{F}$ is a distributive lattice by theorem 381. So we can apply proposition 319. □

The following theorem is essentially borrowed from [18]:

THEOREM 410. Let $\mathfrak{Z}$ be a boolean lattice. Let a be a filter. Then the following are equivalent:

1°. a is prime.
2°. For every $A \in \mathfrak{Z}$ exactly one of $\{A, \overline{A}\}$ is in a.
3°. a is an atom of $\mathfrak{F}$.

PROOF.

1°⇒2°. Let a be prime. Then $A \sqcup^{\mathfrak{Z}} \overline{A} = \top^{\mathfrak{A}} \in a$. Therefore $A \in a \vee \overline{A} \in a$. But since $A \sqcap^{\mathfrak{Z}} \overline{A} = \bot^{\mathfrak{Z}}$ it is impossible $A \in a \wedge \overline{A} \in a$.

2°⇒3°. Obviously $a \neq \bot^{\mathfrak{F}}$. Let a filter $b \sqsubset a$. So $b \supset a$. Let $X \in b \setminus a$. Then $X \notin a$ and thus $\overline{X} \in a$ and consequently $\overline{X} \in b$. So $\bot^{\mathfrak{Z}} = X \sqcap^{\mathfrak{Z}} \overline{X} \in b$ and thus $b = \bot^{\mathfrak{F}}$. So a is atomic.

3°⇒1°. By the previous proposition.

□

4.3.17. Some Criteria.

PROPOSITION 411. Let $\mathfrak{Z}$ be an atomic complete boolean lattice. Then the following conditions are equivalent for any $\mathcal{F} \in \mathfrak{F}$:

1°. $\mathcal{F} \in \mathfrak{P}$;
2°. $\forall S \in \mathscr{P}\mathfrak{F} : \left(\mathcal{F} \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} S \neq \bot \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{F}} \mathcal{K} \neq \bot\right)$;
3°. $\forall S \in \mathscr{P}\mathfrak{P} : \left(\mathcal{F} \sqcap^{\mathfrak{F}} \bigsqcup^{\mathfrak{F}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{F}} K \neq \bot\right)$.

PROOF. The filtrator $(\mathfrak{F};\mathfrak{P})$ is semifiltered by the corollary 362, star separable by 395, with finitely meet-closed core by proposition 364, with separable core by theorem 379. $\mathfrak{P}$ is atomistic because every atomic complete boolean lattice is atomistic. $\mathfrak{F}$ is atomistic by theorem 404.

So we can apply the theorem 320. □

THEOREM 412. If $\mathfrak{Z}$ is a complete boolean lattice then for each $\mathcal{F} \in \mathfrak{F}$FiXme: Too similar to the previous theorem (proposition 4.144)! Also it seems that this theorem can be generalized.

$$\mathcal{F} \in \mathfrak{P} \Leftrightarrow \forall S \in \mathscr{P}\mathfrak{P} : \left(\bigsqcup^{\mathfrak{P}} S \in \partial\mathcal{F} \Rightarrow S \cap \partial\mathcal{F} \neq \emptyset\right).$$

PROOF.

$$\forall S \in \mathscr{P}\mathfrak{P} : \left(\bigsqcup^{\mathfrak{P}} S \in \partial\mathcal{F} \Rightarrow S \cap \partial\mathcal{F} \neq \emptyset\right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{P} : \left(\bigsqcup^{\mathfrak{P}} S \notin \partial\mathcal{F} \Leftarrow S \cap \partial\mathcal{F} = \emptyset\right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{P} : \left(\overline{\bigsqcup^{\mathfrak{P}} S} \in \operatorname{up}\mathcal{F} \Leftarrow \langle\neg\rangle^{*} S \subseteq \mathcal{F}\right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{P} : \left(\bigsqcap^{\mathfrak{P}} S \in \operatorname{up}\mathcal{F} \Leftarrow S \subseteq \operatorname{up}\mathcal{F}\right),$$