4.3.15. More about the Lattice of Filters.

DEFINITION 400. Atoms of $\mathfrak{F}$ (for any poset $\mathfrak{Z}$) are called *ultrafilters*.

DEFINITION 401. Principal ultrafilters are also called *trivial ultrafilters*.

THEOREM 402. If $\mathfrak{Z}$ is a bounded distributive latticeFiXme: Generalize for meet-semilattices? with least element then $\mathfrak{F}$ is an atomic lattice.

PROOF. Let $\mathcal{F} \in \mathfrak{F}$. Let choose (by Kuratowski's lemma) a maximal chain S from $\bot^{\mathfrak{F}}$ to $\mathcal{F}$. Let $S' = S \setminus \{\bot^{\mathfrak{F}}\}$. $a = \bigsqcap^{\mathfrak{F}} S' \neq \bot^{\mathfrak{F}}$ by properties of generalized filter bases (the corollary 389 which uses the fact that $\mathfrak{Z}$ is a distributive lattice with least element). If $a \notin S$ then the chain S can be extended adding there element a because $\bot^{\mathfrak{F}} \sqsubset a \sqsubseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently (taking into account maximality of the chain) there is no $\mathcal{Y} \in \mathfrak{F}$ such that $\bot^{\mathfrak{F}} \sqsubset \mathcal{Y} \sqsubset a$. So a is an atomic filter. Obviously $a \sqsubseteq \mathcal{F}$. □

OBVIOUS 403. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{F}$ is separable.

THEOREM 404. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{F}$ is an atomistic lattice.

PROOF. Because (used the theorem 179) $\mathfrak{F}$ is atomic (theorem 402) and separable. □

COROLLARY 405. If $\mathfrak{Z}$ is a boolean lattice then $\mathfrak{F}$ is atomically separable.

PROOF. By theorem 178. □

THEOREM 406. When $\mathfrak{Z}$ is a boolean lattice, the filtrator $(\mathfrak{F}; \mathfrak{P})$ is central.

PROOF. We can conclude that $\mathfrak{F}$ is atomically separable (the corollary 405), with separable core (the theorem 379), and with join-closed core (corollary 363).

We need to prove $Z(\mathfrak{F}) = \mathfrak{P}$.

Let $\mathcal{X} \in Z(\mathfrak{F})$. Then there exists $\mathcal{Y} \in Z(\mathfrak{F})$ such that $\mathcal{X} \sqcap^{\mathfrak{F}} \mathcal{Y} = \bot^{\mathfrak{F}}$ and $\mathcal{X} \sqcup^{\mathfrak{F}} \mathcal{Y} = \top^{\mathfrak{F}}$. Consequently there is $X \in \operatorname{up} \mathcal{X}$ such that $X \sqcap^{\mathfrak{F}} \mathcal{Y} = \bot^{\mathfrak{F}}$; we also have $X \sqcup^{\mathfrak{F}} \mathcal{Y} = \top^{\mathfrak{F}}$. Suppose $X \sqsupset \mathcal{X}$. Then there exists $a \in \operatorname{atoms}^{\mathfrak{F}} X$ such that $a \notin \operatorname{atoms}^{\mathfrak{F}} \mathcal{X}$. We can conclude also $a \notin \operatorname{atoms}^{\mathfrak{F}} \mathcal{Y}$ (otherwise $X \sqcap^{\mathfrak{F}} \mathcal{Y} \neq \bot^{\mathfrak{F}}$). Thus $a \notin \operatorname{atoms}(\mathcal{X} \sqcup^{\mathfrak{F}} \mathcal{Y})$ and consequently $\mathcal{X} \sqcup^{\mathfrak{F}} \mathcal{Y} \neq \top^{\mathfrak{F}}$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{P}$.

Let now $X \in \mathfrak{P}$. Let $Y = \overline{X}$. We have $X \sqcap^{\mathfrak{P}} Y = \bot^{\mathfrak{F}}$ and $X \sqcup^{\mathfrak{P}} Y = \top^{\mathfrak{F}}$. Thus $X \sqcap^{\mathfrak{F}} Y = \bigsqcap^{\mathfrak{P}} \{X \sqcap^{\mathfrak{P}} Y\} = \bot^{\mathfrak{F}}$; $X \sqcap^{\mathfrak{F}} Y = X \sqcap^{\mathfrak{P}} Y = \top^{\mathfrak{F}}$. We have shown that $X \in (\mathfrak{F})$. □

4.3.16. Atomic Filters.

PROPOSITION 407. If $\mathfrak{Z}$ is a meet-semilattice with least element, then a is an atom of $\mathfrak{P}$ iff $a \in \mathfrak{P}$ and a is an atom of $\mathfrak{F}$.

PROOF. It is semifiltered by the corollary 362, finitely meet-closed by proposition 364. So we can apply the theorem 316. □

PROPOSITION 408. If $\mathfrak{Z}$ is a meet-semilattice with least element then, $a \in \mathfrak{F}$ is an atom of $\mathfrak{F}$ iff $\operatorname{up} a = \partial a$.

PROOF. It is semifiltered by the corollary 362, $\mathfrak{F}$ is a meet-semilattice by the corollary 374. So we can apply theorem 317. □

PROPOSITION 409. If $\mathfrak{Z}$ is bounded distributive lattice, then atomic elements of the filtrator $(\mathfrak{F}; \mathfrak{P})$ are prime.FiXme: Generalize for meet-semilattices?